Rayleigh–Benard convection in a micropolar ferromagnetic fluid

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Abstract

The problem of Rayleigh–Benard convection in a micropolar ferromagnetic fluid layer permeated by a uniform, vertical magnetic field is investigated analytically with free–free, isothermal, spin-vanishing, magnetic boundaries. The influence of the various micropolar and magnetization parameters on the onset of stationary convection has been analysed. It is observed that the micropolar ferromagnetic fluid layer heated from below is more stable as compared with the classical Newtonian ferromagnetic fluid. The nature of influence of the magnetization parameters on convection in the micropolar ferromagnetic fluid is similar to that in the case of Newtonian ferromagnetic fluids. The influence of the micropolar parameters on convection in the ferromagnetic case is akin to its role in the non-magnetic fluid case. The critical wave number is found to be insensitive to the changes in the micropolar fluid parameters, but sensitive to the magnetization parameters. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

The Rayleigh–Benard situation in Eringen’s [1–6] micropolar non-magnetic fluids has been thoroughly investigated by many authors [7–17]. The main result from all these studies is that for heating from below stationary convection is the preferred mode. Eringen’s micropolar fluids are able to describe the behaviour of suspensions, liquid crystals, blood, etc. [18–20]. The corresponding problem in Newtonian ferromagnetic fluids is also well investigated [21–33]. The major assumption in these studies pertaining to magnetic fluids is that the fluid is assumed, on good reasons, to be Newtonian. However, in many situations involving suspensions, as in the magnetic
fluid case, it might be pertinent to demand an Eringen micropolar fluid description. This was suggested in fact by Rosensweig [34] in his monograph. In the light of this suggestion we extend the problem of Rayleigh–Benard convection in Eringen’s micropolar fluid to the micropolar ferromagnetic fluid case with a DC magnetic field. Recently Zahn and Greer [35] have considered interesting possibilities in a planar micropolar ferromagnetic fluid flow with an AC magnetic field.

The problem addressed in the paper has innumerable engineering applications in view of the recent increase in the number of non-isothermal situations wherein magnetic fluids are put to use in place of classical fluids. The monographs of Rosensweig [34] and Bashtovoy et al. [36] review several applications of heat transfer through ferromagnetic fluids.

2. Mathematical formulation and solution

We consider an infinite horizontal layer of a micropolar ferromagnetic fluid layer of depth \( d \) permeated by an externally applied, uniform magnetic field \( H_0 \) normal to the layer. A cartesian coordinate system is taken with the origin in the lower boundary and the \( z \)-axis vertically upwards. The boundaries are assumed to be dynamically free in the absence of surface tension and to be perfect conductors of heat. The no-spin boundary condition is assumed for microration. The boundary condition on the magnetic potential \( \phi \) is \( D\phi = 0 \). This condition on \( \phi \) was also used by Finlayson [21] under the assumption of infinite susceptibility in respect of the perturbed field. The lower and upper boundaries are maintained at constant temperature \( T_0 \) and \( T_1 \) \((< T_0)\), respectively. We adopt the Boussinesq approximation (see [37]) which imposes that the density can be treated as constant everywhere except when multiplied with gravity. The governing equations to be satisfied by the velocity \( \vec{v} \), the pressure \( p \), the microration \( \vec{\omega} \), the magnetic field \( \vec{H} \), the temperature \( T \) and the density \( \rho \) for the thermoconvective problem are:

\[
q_{ij} = 0,
\]

\[
\rho_0 \left[ \frac{\partial q_i}{\partial t} + q_j \frac{\partial q_i}{\partial x_j} \right] = -p_{,i} + \rho g_i + \frac{1}{2}\zeta \left( \dot{q}_{ij} - \theta \right) q_{ij} + \zeta \vec{e}_{ij,k} \omega_{k,j} + \left( H_i B_j \right)_{,j},
\]

\[
\rho_0 \left[ \frac{\partial \omega_i}{\partial t} + q_j \frac{\partial \omega_i}{\partial x_j} \right] = \zeta (\dot{e}_{ijk} q_k - 2 \omega_i) + \mu_0 \vec{e}_{ijk} M_j H_k + \left( \lambda + \eta' \right) \omega_{k,kl} + \eta' \omega_{i,jj},
\]

\[
\frac{\partial T}{\partial t} + q_j \frac{\partial T}{\partial x_j} = \frac{\delta}{\rho_0 c_v} T_j \vec{e}_{ijk} \omega_{k,j} + K_{,j},
\]

\[
\rho = \rho_0 [1 - \alpha (T - T_0)].
\]

Maxwell’s equations, simplified for a non-conducting fluid with no displacement currents, become

\[
B_{ij} = 0, \quad \epsilon_{imm} H_{n,m} = 0,
\]

\[
B_i = \mu_0 (H_i + M_i).
\]

We assume that the magnetization is aligned with the magnetic field, but allows a dependence on the magnitude of the magnetic field as well as the temperature
\[ M_r(H, T) = \frac{H}{H} M(H, T). \] (8)

The magnetic equation of state is linearized about the magnetic field \( H_0 \) and an average temperature \( T_0 \), to become
\[ M(H, T) = M_0 + \chi(H - H_0) - K_m(T - T_0). \] (9)

In the above equations, the pressure \( p \) consists of the normal hydrostatic stress and a magnetic stress \((\mu_0 H^2)/2\), \( g \) is the acceleration due to gravity, \( \eta \) is the shear kinematic viscosity coefficient, \( \lambda' \) and \( \eta' \) are the bulk and shear spin viscosity coefficients, \( \zeta \) is the coupling viscosity coefficient or vortex viscosity, \( \delta \) is the micropolar heat conduction coefficient, \( C_v \) is the specific heat, \( \alpha \) is the coefficient of thermal expansion, \( \rho_0 \) is the density of the fluid at temperature \( T = T_0 \), \( I \) is the moment of inertia, \( K_c \) is the thermal diffusivity, \( \chi \) is the susceptibility and \( K_m \) is the pyromagnetic coefficient.

It is possible to have the collinearity of magnetic field with fluid magnetization as in Eq. (8), since we are considering only DC fields. For an AC magnetic field the magnetization constitutive equation will be different from Eq. (9) and shall have to be a modified form of the one taken by [35] to include thermal effects. The magnetization is thus aligned with the DC magnetic field considered in the paper and hence there is no body couple in Eq. (3).

2.1. Basic state

The basic state is one in which
\[ \bar{q}_b = (0, 0, 0), \quad \bar{\omega}_b = (0, 0, 0), \quad p = p_b(z), \quad \rho = \rho_b(z), \quad \bar{H}_b = H_b\hat{k}, \]
\[ \bar{M}_b = M_b\hat{k}, \quad T_b = T_b(z), \] (10)

where the subscript \( b \) denotes the basic state.

Substituting Eq. (10) into Eqs. (2), (4) and (5), we get the basic state governing equations as:
\[ (p_b)_{ii} = \rho_b g_i + (B_b)_k (H_b)_{k,i}, \] (11)
\[ 0 = \frac{d^2 T_b}{dz^2}, \] (12)
\[ \rho_b = \rho_0 [1 - \alpha(T_b - T_0)]. \] (13)

Substituting Eq. (10) into Eq. (6) and noting that \( B_b = \mu_0 (H_b + M_b) \), we obtain
\[ \frac{\partial}{\partial z} (H_b + M_b) = 0, \]
which yields
\[ M_b(z) + H_b(z) = C_1, \] (14)

where \( C_1 \) is the constant of integration.
Also

\[
T_b = T_0 - \left( \frac{\Delta T}{d} \right) z. \tag{15}
\]

Substituting Eq. (10) into Eq. (9) yields

\[
M_b = M_0 + \chi (H_b - H_0) - K_m (T_b - T_0). \tag{16}
\]

Substituting Eqs. (14) and (15) into Eq. (16) and simplifying give Eqs. (17) and (18) [21]:

\[
\tilde{H}_b = \left[ H_0 - \frac{K_m}{(1 + \chi)} \left( \frac{\Delta T}{d} \right) z \right] \hat{k}, \tag{17}
\]

\[
\tilde{M}_b = \left[ M_0 + \frac{K_m}{(1 + \chi)} \left( \frac{\Delta T}{d} \right) z \right] \hat{k}. \tag{18}
\]

Let the basic state be slightly perturbed. We now have

\[
\tilde{q} = q_b + \tilde{q}', \quad \tilde{\sigma} = \tilde{\sigma}_b + \tilde{\sigma}', \quad p = p_b + p', \quad \rho = \rho_b + \rho', \quad T = T_b + T',
\]

\[
\tilde{H} = H_b \hat{k} + \tilde{H}', \quad \tilde{M} = M_b \hat{k} + \tilde{M}'. \tag{19}
\]

The prime indicates that the quantities are infinitesimal perturbations.

Substituting Eq. (19) into Eqs. (8) and (9) and assuming \( K_m (\Delta T / d) z \ll (1 + \chi) H_0 \), we get [21]:

\[
M_i' = \chi H_i' - K_m T_i', \tag{20}
\]

\[
M_i' + H_i' = \left( 1 + \frac{M_0}{H_0} \right) H_i', \quad i = 1, 2. \tag{21}
\]

From Eq. (20), we obtain

\[
M_3' = \chi H_3' - K_m T_3'. \tag{22}
\]

Hence, using Eq. (22), we get

\[
M_3' + H_3' = \chi H_3' - K_m T_3' + H_3' = (1 + \chi) H_3' - K_m T_3'. \tag{23}
\]

Substituting Eq. (19) into Eqs. (2)–(4) and taking curl twice on the first equation and once on the second equation and considering only the \( k \)th component, and substituting Eq. (6) in Eq. (23), we get:
\[-\rho_0 \frac{\partial}{\partial t} (w'_{jj}) = -\rho_0 x g (T'_{jj} - T',_{33}) - \frac{\mu_0 K_m^2}{(1 + \chi)} \frac{\Delta T}{d} (T'_{jj} - T',_{33})
\]

\[+ \mu_0 K_m \frac{\Delta T}{d} (\phi_{jj} - \phi_{33})_{,3} - \zeta \Omega_{3, jj} - (2 \zeta + \eta) w'_{jj},\]

\[\rho_0 I \frac{\partial}{\partial t} (\Omega_3) = -\zeta \left[ w'_{jj} + 2 \Omega_3 \right] + \eta \Omega_3_{,jj},\]

\[\frac{\partial T'}{\partial t} = K_c T'_{jj} + \left( -\frac{\Delta T}{d} \right) \left( w' - \frac{\delta \Omega_3}{\rho_0 C_v} \right),\]

\[\left( 1 + \frac{M_0}{H_0} \right) (\phi_{jj} - \phi_{33}) + (1 + \chi) \phi_{33} - K_m T'_3 = 0,\]

where

\[\Omega_3 = (\nabla \times \vec{\omega})_3, \quad \vec{q} = (u, v, w), \quad H'_3 = \frac{\partial \phi}{\partial z}.\]

In the case of Eringen’s micropolar fluids it is now well known that stationary convection is the preferred mode when the micropolar fluid layer is heated from below. Also, when heated from above the convection does set in as oscillatory motions but only at very large values of the Rayleigh number, which is perhaps unrealistic. In the present problem involving Eringen’s micropolar ferromagnetic fluid also the story is no different and we assume the principle of exchange of stability to be valid and deal with only stationary convection. Hence time derivatives will henceforth be dropped.

The perturbation equations (24)–(27) are non-dimensionalized using the following definitions:

\[\left( x^*, y^*, z^* \right) = \left( \frac{x}{d}, \frac{y}{d}, \frac{z}{d} \right), \quad w^* = \frac{w}{(K_c/d)}, \quad T^* = \frac{T'}{\Delta T}, \quad \Omega_3^* = \frac{\Omega_3}{(K_c/d^3)},\]

\[\phi^* = \frac{\phi}{(K_m K_c (\zeta + \eta) R) / (\rho_0 x g d^2 (1 + \chi))}.\]

Non-dimensionalizing Eqs. (24)–(27), we get:

\[(1 + N_1)w_{jj} + N_1 \Omega_3_{,jj} + R(1 + M_1) (T_{jj} - T_{33}) - M_1 R (\phi_{jj} - \phi_{33})_{,3} = 0,\]

\[N_3 \Omega_{3, jj} - N_1 w_{jj} - 2 N_1 \Omega_3 = 0,\]

\[T_{jj} + (w - N_3 \Omega_3) = 0,\]

\[M_3 (\phi_{jj} - \phi_{33}) + \phi_{33} - T_3 = 0,\]

\[(1 + N_1)w_{jj} + N_1 \Omega_3_{,jj} + R(1 + M_1) (T_{jj} - T_{33}) - M_1 R (\phi_{jj} - \phi_{33})_{,3} = 0,\]

\[N_3 \Omega_{3, jj} - N_1 w_{jj} - 2 N_1 \Omega_3 = 0,\]

\[T_{jj} + (w - N_3 \Omega_3) = 0,\]

\[M_3 (\phi_{jj} - \phi_{33}) + \phi_{33} - T_3 = 0,\]
where the asterisks have been dropped for simplicity and

\[
N_1 = \frac{\zeta}{\eta + \zeta} \quad \text{(coupling parameter)},
\]

\[
N_3 = \frac{\eta'}{(\eta + \zeta)d^2} \quad \text{(couple stress parameter)},
\]

\[
N_2 = \frac{\delta}{\rho_0 C_c d^2} \quad \text{(micropolar heat conduction parameter)},
\]

\[
R = \frac{\rho_0 \alpha g \Delta T d^3}{(\eta + \zeta)K_c} \quad \text{(Rayleigh number)},
\]

\[
M_1 = \frac{\mu_0 K^2_m \Delta T}{z g (1 + \zeta) \rho_0 d} \quad \text{(magnetization parameter [20])},
\]

\[
M_3 = \frac{(1 + M_0 \eta_0)}{(1 + \zeta)} \quad \text{(magnetization parameter [20])}.
\]

If \( \zeta = 0 \), then the above Rayleigh number identifies itself with the classical definition.

Eqs. (29)–(32) are solved subject to the following boundary conditions appropriate for stress free isothermal, no-spin boundaries, and infinite magnetic susceptibility with respect to the perturbed field.

\[
w = \frac{\partial^2 w}{\partial z^2} = T = \Omega_3 = D \phi = 0 \quad \text{at } z = 0, 1.
\]

(33)

In accordance with the requirement of the normal mode analysis for stationary convection we assume that the perturbations are periodic waves and take [37]

\[
[w, T, \Omega_3, \phi] = [W(z), \theta(z), G(z), \phi(z)] \exp \{i(lx + my)\},
\]

(34)

where \( l \) and \( m \) are the \( x \) and \( y \) components of the horizontal wave number \( \tilde{a} \).

Substituting Eq. (34) into Eqs. (29)–(32) and eliminating \( G(z) \), \( \theta(z) \) and \( \phi(z) \) between the resulting equations, we get a single equation involving \( W(z) \) in the form

\[
R(1 + M_1)a^2(D^2 - M_3a^2)[N_3(D^2 - a^2) - 2N_1 - N_1N_3(D^2 - a^2)]W
+ M_1 Ra^2D^2[N_1N_3(D^2 - a^2) - N_3(D^2 - a^2) + 2N_1]W
+ (D^2 - a^2)^3(D^2 - M_3a^2)[N_1^2 + (1 + N_1)(N_3(D^2 - a^2) - 2N_1)]W = 0,
\]

(35)

where \( D = d/\partial z \).

The solution for \( W \) for the lowest mode is as per Finlayson [21], taken in the form

\[
W = A \sin \pi z,
\]

(36)

where \( A \) is a constant. Substitution of solution (36) in Eq. (35) leads to a characteristic equation which on rearrangement yields
where \( K^2 = \pi^2 + a^2 \). Eq. (37) is quite a general one for the eigenvalue \( R \). The classical results in respect of Newtonian ferromagnetic fluids and micropolar non-magnetic fluids can be obtained as limiting cases of the present study.

Setting \( N_1 = 0 \) and keeping \( N_3 \) and \( N_5 \) arbitrary in Eq. (37), we get

\[
R = \frac{K^6}{a^2 \left[ 1 + M_1 - \frac{M_1 \pi^2}{(\pi^2 + M_3 a^2)} \right]},
\]

the classical Rayleigh–Benard result as in [21] for the Newtonian ferromagnetic fluid case.

Setting \( M_1 = 0 \) in Eq. (37), we get

\[
R = \frac{K^6 \left[ (1 + N_1)N_3 K^2 + 2N_1 + N_1^2 \right]}{a^2 \left[ (N_3 - N_5 N_1)K^2 + 2N_1 \right]},
\]

which is the expression for the Rayleigh number of micropolar non-magnetic fluids, discussed by [9,10,13] and others.

Setting \( N_1 = 0 \) and keeping \( N_3 \) and \( N_5 \) arbitrary in Eq. (39), we get

\[
R = \frac{K^6}{a^2},
\]

the classical Rayleigh–Benard result.

3. Results and discussion

The sensitiveness of the critical Rayleigh number \( R_c \) to the changes in the magnetization parameters \( M_1, M_3 \) and to the micropolar fluid parameters \( N_1, N_3 \) and \( N_5 \) is depicted in Figs. 1–5. The simplest boundary conditions chosen in the paper, namely free–free, no-spin, isothermal with infinite susceptibility in the perturbed field keep the problem analytically tractable and serve the purpose of providing a qualitative insight into the problem. Furthermore, the choice of a DC magnetic field makes life simpler but nevertheless serves the purpose stated earlier.

Figs. 1 and 2 are plots of \( R_c \) versus the magnetization parameters \( M_1 \) and \( M_3 \). \( M_1 \) is a ratio of the magnetic to gravitational forces. As \( M_1 \) increases \( R_c \) increases. Hence \( M_1 \) has a stabilizing effect on the fluid. \( M_1 \) represents the departure of the magnetic equation of state from linearity. As the equation of state becomes more non-linear (\( M_1 \) large) the fluid layer is destabilized slightly.

Fig. 3 is a plot of \( R_c \) versus the coupling parameter \( N_1 \). Clearly \( R_c \) increases with increasing \( N_1 \). Increase in \( N_1 \) indicates the increasing concentration of microelements. Since microelements increase in number with increasing \( N_1 \), a greater part of the energy of the system is consumed by these elements in developing gyrrational velocities of the fluid, and as a result onset of convection is delayed.
Fig. 4 is a plot of $R_c$ versus couple stress parameter $N_3$. Clearly $R_c$ decreases with increasing $N_3$ and ultimately levels off to the Newtonian value. Increase in $N_3$ increases the couple stress of the fluid which causes a decrease in microrotation and hence makes the system more unstable. At only small values of $N_3$ couple stresses are operative and hence we observe that microrotations (small values of $N_3$) stabilize the system in comparison with the Newtonian problem.

Fig. 1. Plot of critical Rayleigh number $R_c$ vs. $M_1$ for various values of $M_3$.

Fig. 2. Plot of $R_c$ vs. $M_3$ for various values of $M_1$. 
Fig. 3. Plot of $R_c$ vs. the coupling parameter $N_1$ for various values of $N_5$.

Fig. 4. Plot of $R_c$ vs. the couple stress parameter $N_3$ for various values of $N_1$.

Fig. 5 is a plot of $R_c$ versus the micropolar heat conduction parameter $N_5$. When $N_5$ increases, the heat induced into the fluid due to the microelements is also increased, thus reducing the heat transfer from the bottom to the top. The decrease in heat transfer is responsible for delaying the
onset of instability. Thus increasing $N_5$ leads to increase in $R_c$. In other words $N_5$ stabilizes the flow.

Table 1 shows the influence of magnetization and micropolar parameters on the critical wave number $a_c$. Clearly the critical wave number is, in general, insensitive to the changes in the micropolar parameters but sensitive to changes in the magnetization parameters. Large magnetization parameters help in inducing the coupling number $N_1$ into influencing $a_c^2$.

In view of the increasing number of ferromagnetic fluid applications involving AC magnetic field, there is a need to generalize the above problem to incorporate the non-zero body torque in

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the conservation of angular momentum and a thermal term in the magnetization constitutive equation as taken by Zahn and Greer [35]. This highly involved work is presently being attempted.

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