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Cross-diffusion and viscoelastic effects on multidiffusive porous convection

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Abstract

The onset of triply cross-diffusive convection in a viscoelastic fluid-saturated porous layer is investigated as the study is found very relevant for describing natural phenomena (contaminant transport, underground water flow, improved oil recovery, polymer processing). A modified Darcy-Oldroyd-B model is used to describe the viscoelastic fluid flow in a porous medium with full cross-diffusion terms in the diffusivity matrix. A normal mode analysis yields an exact dispersion equation of fifth degree and accordingly the criterion for the onset of stationary and oscillatory convection is obtained. The numerical computations are carried out for diffusivity elements experimentally determined for lysozyme-sodium chloride-bovine serum albumin (BSA)-water system. Instability is found to occur via oscillatory mode for a certain choice of governing parameters. The relaxation and retardation viscoelastic parameters portray opposing contributions on the oscillatory onset and an increase in the relaxation parameter is to increase the range of retardation parameter up to which the oscillatory convection is preferred. The cross-diffusion is to either delay/hasten the onset of instability based on the magnitude of the stratifying agents. Even minute variations in the cross-diffusion elements indict complete change in the linear instability criteria. The topology of neutral curves disclosed the occurrence

of disconnected closed convex oscillatory neutral curve revealing the requirement of three critical solute Darcy-Rayleigh numbers to state fully the instability criteria instead of the usual single value; a novel result ensured from the study. Moreover, the nature of instability for Oldroyd-B, Maxwell and Newtonian fluids turns out to be dissimilar for the same governing parameters.

KEYWORDS

cross-diffusion, disconnected neutral curve, Oldroyd-B fluid, porous media, triple-diffusive convection

1 | INTRODUCTION

Many technologically important alloys contain significant mass fractions of three or more metallic elements. When ingots of these materials are grown by directional solidification using cooling processes, the rejection of solutes at the growing interface leads to the development of concentration gradients. If the solutes have higher partial molar densities than the bulk melt, this configuration may be convectively unstable. In such circumstances, convection may be important in the interdendritic mushy zone-regarded as a porous medium. In this sense, the potential importance of triple or multidiffusive convection in a porous medium is clear. Another application of triple-diffusive convection can be found in experiments on double-diffusive porous convection in which the effect of dyes or small temperature gradients should be considered. Further applications may be found in the area of contaminant transport (Celia et al¹; Chen et al²) where the chemical species that form the contaminants are nonreactive.

The diffusion of any species in liquid mixtures depends merely on its concentration gradient rather than on the spatial distribution of other species. Although the diffusion mechanism seems to be a simple process in single-component systems, in multicomponent systems, a range of possibilities becomes obvious when the diffusion process involves the contribution of two or more diffusive agents. In such cases, variety of phenomena like self-diffusion, intradiffusion, interdiffusion, tracer diffusion, uphill diffusion, mutual diffusion, and cross-diffusion are bound to occur. The cross-diffusion is a phenomenon in which the concentration gradient of one species induces a flux of the other species. In the absence of chemical reactions, the cross-diffusion induces convective motions around liquid interfaces. On the other hand, cross-diffusion is also responsible for processes like chemotaxis, weakly nonbonding solute-solute interactions, electrostatic, etc. The possibility of cross-diffusion terms in a multicomponent system was suggested by Onsager and Fuoss,³ while Baldwin et al⁴ undertook the experimental verification of the existence of cross-diffusion and also observed that cross-diffusion coefficients can be quite significant.

The motion of particles in the solution mainly depends on the magnitude of diffusion coefficients, which in turn depends on the composition of solution and solute concentration gradients. For determining the diffusion coefficients, a suitable frame of reference is essential. The solvent-fixed reference frame, the mass-fixed reference frame, and the volume-fixed reference frame are three main reference frames, out of which the last one is found to be most suitable for laboratory experiments.

When the concentrations are low, the interactions between particles become negligible, then the self-diffusion occurs and in the diffusion coefficients matrix, the diagonal elements tend to be self-diffusion coefficients. These coefficients do not depend on the choice of the reference frame. The relationship between self and cross-diffusion coefficients was first determined by Mimura and Kawasaki.⁵ In any fluid system with N -species, the diffusion processes are described by the generalized Fick's law, which was developed by Adolf Fick in the 19th century

$$\vec{F}_m = - \sum_{n=1}^N D_{mn} \nabla S_n,$$

which indicates that the flux $\vec{F}_m$ of m th species depends on the concentration gradients (∇S_n) of all the species and D_{mn} are the diffusion coefficients matrix of component m into component n . For $m = n$, the diagonal elements D_{mm} of the diffusion matrix represent the self-diffusion coefficients, while the off-diagonal elements D_{mn} corresponding to $m \neq n$ specify the cross-diffusion coefficients. The motion of a species due to its concentration gradient leads to the flux of the other species either along or against its direction of motion. Based on this, the cross-diffusion coefficients can either be positive (co-flux) or negative (counter-flux). The cross-diffusion terms in the diffusivity matrix control the instability of the system considerably but they are commonly unnoticed on the basis of the often repeated slogan that these are of smaller magnitude compared with the main diagonal elements. But in some liquid mixtures, cross-diffusion terms (off-diagonal elements) are found to be much larger than the self-diffusion (main diagonal elements) terms.^{6,7}

The phenomenon of triple-diffusive convection in a porous layer, whereby three competing stratifying agencies affect the density distribution is of paramount importance. The dynamics of the triple-diffusive system may be more interesting than that of the double-diffusive situation in that new convective phenomena may occur because of the presence of additional diffusing component which may be either stabilizing or destabilizing. By adopting the formalism of Griffiths,⁸ Rudraiah and Vortmeyer⁹ and Poulikakos¹⁰ investigated the effect of the third diffusing component on double-diffusive convection in a porous medium. A systematic study of linear and nonlinear triple-diffusive convection in a porous layer was later considered by Tracey¹¹ and disclosed unexpected instability characteristics in the vein of Pearlstein et al¹² who showed that the results obtained by Griffiths⁸ are incomplete. For triple-diffusive convection in porous media, Rionero^{13,14} obtained the conditions sufficient for the onset of convection and the absence of subcritical instabilities. In all the above-mentioned studies, cross-diffusion effects are ignored.

Of late, interest in non-Newtonian fluid flows through porous media has grown significantly as this situation occurs in more than a few applications such as improved oil recovery, filtration, liquid complex molding, polymer processing, among others.¹⁵ But it is hard to find only one model that displays all behaviors of non-Newtonian fluids. Consequently, non-Newtonian fluids with different constitutive models have been used depending on the circumstances and in particular viscoelastic fluid theories are used extensively. Many investigators have applied the Oldroyd-B fluid theory in dealing with viscoelastic instability in dilute polymer solutions which can be highly elastic while maintaining essentially a constant viscosity.¹⁶⁻¹⁸ Thermal convective instability in a viscoelastic fluid-saturated porous layer has been investigated extensively.¹⁹⁻²⁵ Attention is also focused on the studies of double-diffusive convection in a viscoelastic fluid-saturated porous layer.²⁶⁻³⁰ However, the study of triple-diffusive convection in viscoelastic fluids in porous media is still in infancy despite its relevance in diverse fields as mentioned in the beginning. Triple-diffusive convection in a Maxwell fluid-saturated porous medium was considered by Zhao et al,³¹ while Raghunatha et al³² revisited this study and

disclosed some unusual results, the system is capable of supporting by performing both steady and unsteady stability analyses and they also quantified the amount of heat and mass transfer through Nusselt numbers. Recently, Kalita and Karmakar³³ systematically carried out a semi-analytic investigation to see the nonlinear evolutionary dynamics of self-gravitating viscoelastic nonthermal polytropic astrofluid on the Jeansian scales of space and time. Whereas Karmakar and Kalita³⁴ reported a new generalized viscoelastic fluid model to investigate the gravitational instability in the presence of realistic astronomical relevancy.

As envisaged, it is recognizable that the cross-diffusion effects are bound to occur when more than one compositional gradient is present. In some liquid mixtures, cross-diffusion terms are found to be much larger than self-diffusion terms.^{6,7} The inevitability of off-diagonal elements (cross-diffusion terms) in ternary/quaternary system was recommended by Onsager and Fuoss.³ Baldwin et al.² conducted the experimental corroboration of the existence and importance of cross-diffusion (off-diagonal elements) effects. Mimura and Kawasaki⁵ explained the relations between self- and cross-diffusion terms. The studies conducted on double-diffusive convection with cross-diffusion effects in a fluid-saturated porous medium are mainly limited to Newtonian fluids.³⁵⁻³⁹ The effect of cross-diffusion on the onset of gravitational instability in a Hele-Shaw cell saturated with ternary solution considering all cross-diffusion coefficients was theoretically analyzed by Kim and Song,⁴⁰ while Kim and Song⁴¹ studied both theoretically and numerically the effect of cross-diffusion on the onset and the growth of the gravitational instability of a miscible interface. But, a limited number of studies analyzed the effect of cross-diffusion on double-diffusive convection in non-Newtonian fluid-saturated porous media. Awad et al.⁴² considered the effect of cross-diffusion on the onset of thermosolutal convection using the Darcy-Brinkman-Maxwell model. Wang and Tan²⁸ studied the Soret effect on the stability of viscoelastic thermosolutal convection in a fluid-saturated porous medium, while Malashetty and Biradar⁴³ investigated the problem considering both Soret and Dufour effects.

It is intriguing to note that the existence of cross-diffusion in the diffusivity matrix contributes significantly to the instability of the system and even alters the character of the instability mechanism. The intricacies of cross-diffusion on convective instability in a triply diffusive Newtonian fluid layer have been deliberated by Terrones⁴⁴ and in an Oldroyd-B fluid layer by Raghunatha et al.⁴⁵ Nonetheless, off-diagonal contributions on the onset of triple-diffusive convection in a porous layer have not been explored despite recognizing the ramifications of cross-diffusion terms on the instability characteristics. We have examined the effect of all nine elements in the diffusivity matrix on the instability of the viscoelastic triply diffusive fluid-saturated porous layer for the first time in this article. A modified Darcy-Oldroyd-B model is used to analyze the flow in a porous medium. The study is more general in nature and the results of Maxwell and the Newtonian fluids are delineated as particular cases from the present study.

2 | MATHEMATICAL FORMULATION

We consider a viscoelastic incompressible fluid-saturated horizontal layer of Darcy porous medium of width d . The coordinates are selected with positive z -axis vertically upward and positive x -axis in the horizontal direction. The fluid density ρ depends on three different stratifying agents having different solute diffusivities and assumed to change linearly with solute concentrations in the form

$$\rho = \rho_0 \left[1 + \sum_{m=1}^3 \alpha_m (S_m - S_{m0}) \right], \quad (1)$$

where S_m ($m = 1, 2, 3$) and α_m are the solute concentration and the expansion coefficient of the m th stratifying agency, respectively, and ρ_0 is the reference density at the reference solute concentration S_{m0} . By analogy with Oldroyd-B constitutive relationships, a phenomenological model, which relates pressure drop and velocity for an incompressible viscoelastic fluid in a porous medium was first given by Alishaev and Mirzadjanzade⁴⁶ and Khuzhayorov et al,¹⁶ which can be generalized for a Boussinesq fluid in the form²⁴

$$\nabla \cdot \vec{q} = 0 \quad (2a)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \left(\frac{\rho_0}{\varepsilon} \frac{\partial \vec{q}}{\partial t} + \nabla p - \rho \vec{g} \right) = -\frac{\mu}{K} \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \vec{q}, \quad (2b)$$

where $\vec{q} = (u, v, w)$ is the velocity, μ is the fluid viscosity, p is the pressure, ε is the porosity, K is the permeability of the porous medium, λ_1 is the stress relaxation time, and λ_2 is the strain retardation time. Equation (2) simplifies to Darcy's law for flow of viscous Newtonian fluid through a porous medium when $\lambda_1 = 0 = \lambda_2$. The conservation equation is applied to each stratifying agent in the mixture considered to be incompressible (Nield and Bejan⁴⁷). The conservation equations for the stratifying agencies are

$$\frac{DS_m}{Dt} = -\nabla \cdot \vec{F}_m \quad (m = 1, 2, 3), \quad (3a)$$

where $D/Dt = \partial/\partial t + (\vec{V} \cdot \nabla)$ and $\vec{V}$ is the intrinsic average velocity. Substituting for $\vec{F}_m$ and using the Dupuit-Forchheimer relationship $\vec{q} = \varepsilon \vec{V}$, we get (for three stratifying agencies)

$$\varepsilon \frac{\partial S_m}{\partial t} + \nabla \cdot (\vec{q} S_m) = \sum_{n=1}^3 \nabla \cdot (D_{mn} \nabla S_n) \quad (m = 1, 2, 3), \quad (3b)$$

where D_{mn} ($m, n = 1, 2, 3$) are the effective solute diffusivities. In (3b), one of the stratifying agencies could be temperature also and then the definition of ε changes and it represents the ratio of heat capacities.

The boundaries are impermeable and perfect conductors of stratifying agencies. The relevant boundary conditions are

$$\vec{q} \cdot \hat{k} = 0 \text{ at } z = 0, d.$$

$$S_m = S_{m0} \quad (m = 1, 2, 3) \text{ at } z = 0 \text{ and } S_m = S_{m0} - \Delta S_m \quad (m = 1, 2, 3) \text{ at } z = d, \quad (4)$$

where $\Delta S_m > 0$. The above-stated governing nonlinear partial differential equations are rendered to dimensionless form using the transformations

$$\nabla^* = d \nabla, \quad \vec{q}^* = \frac{d}{D_{11}} \vec{q}, \quad p^* = \frac{K}{\mu D_{11}} p, \quad t^* = \frac{D_{11}}{d^2 \varepsilon} t, \quad S_m^* = \frac{(S_m - S_{m0})}{\Delta S_m} \quad (m = 1, 2, 3). \quad (5)$$

Substituting (5) in (2) and (3), we respectively obtain

$$\nabla \cdot \vec{q}^* = 0 \quad (6a)$$

$$\left(1 + \Lambda_1 \frac{\partial}{\partial t}\right) \left[\frac{1}{Pr_D} \frac{\partial \vec{q}}{\partial t} + \nabla p + (R_1 S_1 + R_2 S_2 + R_3 S_3) \hat{k} \right] = - \left(1 + \Lambda_2 \frac{\partial}{\partial t}\right) \vec{q}, \quad (6b)$$

$$\frac{\partial S_m}{\partial t} + \nabla \cdot (\vec{q} S_m) = \sum_{n=1}^3 \nabla \cdot \left(\gamma_{mn} \frac{R_n}{R_m} \nabla S_n \right) \quad (m = 1, 2, 3), \quad (7)$$

where $\Lambda_1 = \lambda_1 D_{11}/d^2 \varepsilon$ is the viscoelastic stress relaxation time parameter, $\Lambda_2 = \lambda_2 D_{11}/d^2 \varepsilon$ is the viscoelastic strain retardation time parameter, $R_m = \alpha_m g \Delta S_m K d / \nu D_{11}$ ($m = 1, 2, 3$) are the solute Darcy-Rayleigh numbers, $\gamma_{mn} = \frac{\alpha_m D_{mn}}{\alpha_n D_{11}}$ ($m, n = 1, 2, 3$) are the ratios of diffusivities, and $Pr_D = \nu d^2 \varepsilon^2 / D_{11} K$ is the Darcy-Prandtl number.

The fundamental state is quiescent, and the solute concentrations, as well as pressure distributions, are found to be

$$S_{mb}(z) = -z \quad (m = 1, 2, 3), \quad p_b(z) = p_0 + (R_1 + R_2 + R_3) z^2 / 2. \quad (8)$$

To examine the onset of convection, we superimpose small perturbations on the basic state of the form

$$\vec{q} = 0 + \vec{q}', \quad p = p_b(z) + p', \quad S_m = S_{mb}(z) + S'_m \quad (m = 1, 2, 3). \quad (9)$$

Substituting (9) into (6) and (7), eliminating the pressure term from (6) by using cross-differentiation and introducing a stream function $\psi'(x, z, t)$ through

$$u' = \frac{\partial \psi'}{\partial z} \quad \text{and} \quad w' = -\frac{\partial \psi'}{\partial x}, \quad (10)$$

we note that the governing stability equations (6) and (7), respectively, become

$$\left(1 + \Lambda_1 \frac{\partial}{\partial t}\right) \left[\frac{1}{Pr_D} \frac{\partial}{\partial t} (\nabla^2 \psi) - R_1 \frac{\partial S_1}{\partial x} - R_2 \frac{\partial S_2}{\partial x} - R_3 \frac{\partial S_3}{\partial x} \right] = - \left(1 + \Lambda_2 \frac{\partial}{\partial t}\right) (\nabla^2 \psi), \quad (11)$$

$$\frac{\partial}{\partial t} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \frac{1}{R_1} \begin{bmatrix} 1 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \nabla^2 \begin{bmatrix} R_1 S_1 \\ R_2 S_2 \\ R_3 S_3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{\partial \psi}{\partial x} + \begin{bmatrix} J(\psi, S_1) \\ J(\psi, S_2) \\ J(\psi, S_3) \end{bmatrix}, \quad (12)$$

where $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial z^2$ and $J(\cdot, \cdot)$ represents the Jacobian with respect to x and z .

The boundary conditions now become

$$\psi = S_1 = S_2 = S_3 = 0 \quad \text{at} \quad z = 0, 1. \quad (13)$$

3 | LINEAR INSTABILITY ANALYSIS

Equations (11) and (12) are linearized by neglecting the nonlinear terms and a normal mode solution is assumed in the form

$$\psi = A e^{\sigma t} \sin \alpha x \sin \pi z, \quad S_m = B_m e^{\sigma t} \cos \alpha x \sin \pi z \quad (m = 1, 2, 3), \quad (14)$$

where A and $B_1 - B_3$ are constants, α is the horizontal wave number, and σ is the growth term. Substituting (14) in the linearized versions of (11) and (12), we get a set of homogeneous algebraic equations, which can be written in the matrix form as

$$\begin{bmatrix} \frac{\sigma\delta^2}{Pr_D} + \frac{(1 + \Lambda_2\sigma)}{(1 + \Lambda_1\sigma)}\delta^2 & -\alpha R_1 & -\alpha R_2 & -\alpha R_3 \\ \alpha & \sigma + \gamma_{11}\delta^2 & \gamma_{12}\frac{R_2}{R_1}\delta^2 & \gamma_{13}\frac{R_3}{R_1}\delta^2 \\ \alpha & \gamma_{21}\frac{R_1}{R_2}\delta^2 & \sigma + \gamma_{22}\delta^2 & \gamma_{23}\frac{R_3}{R_2}\delta^2 \\ \alpha & \gamma_{31}\frac{R_1}{R_3}\delta^2 & \gamma_{32}\frac{R_2}{R_3}\delta^2 & \sigma + \gamma_{33}\delta^2 \end{bmatrix} \begin{bmatrix} A \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (15)$$

where $\delta^2 = \alpha^2 + \pi^2$. For the occurrence of nonzero solution, the determinant of the coefficient matrix of (15) should be zero and which yields the characteristic equation

$$a_1 \sigma^5 + a_2 \sigma^4 + a_3 \sigma^3 + a_4 \sigma^2 + a_5 \sigma + a_6 = 0, \quad (16)$$

where

$$\begin{aligned} a_1 &= \delta^2 \Lambda_1; \quad a_2 = \delta^2 + \delta^2 Pr_D \Lambda_2 + \delta^4 \Lambda_1 \sum_{i=1}^3 \gamma_{ii}, \\ a_3 &= Pr_D \left[\delta^2 + \alpha^2 \Lambda_1 \sum_{i=1}^3 R_i \right] + \delta^4 (1 + Pr_D \Lambda_2) \sum_{i=1}^3 \gamma_{ii} + \delta^6 \Lambda_1 b_7, \\ a_4 &= \alpha^2 Pr_D \sum_{i=1}^3 b_i R_i + b_4 \delta^4 + b_5 \delta^6 + b_6 \delta^8; \quad a_5 = Pr_D \alpha^2 \delta^2 \left[\sum_{i=1}^3 c_i R_i \right] + c_4 \delta^6 + c_5 \delta^8, \\ a_6 &= Pr_D \alpha^2 \delta^4 \sum_{i=1}^3 d_i R_i + Pr_D \delta^8 d_4, \end{aligned}$$

with

$$\begin{aligned} b_1 &= 1 + \delta^2 \Lambda_1 [\gamma_{33} + \gamma_{22} - \gamma_{12} - \gamma_{13}]; \quad b_2 = 1 + \delta^2 \Lambda_1 [\gamma_{11} + \gamma_{33} - \gamma_{23} - \gamma_{21}], \\ b_3 &= 1 + \delta^2 \Lambda_1 [\gamma_{22} + \gamma_{11} - \gamma_{31} - \gamma_{32}]; \quad b_4 = Pr_D \sum_{i=1}^3 \gamma_{ii}; \quad b_5 = b_7 (1 + Pr_D \Lambda_2); \quad b_6 = \Lambda_1 d_4, \\ b_7 &= \gamma_{11} \gamma_{22} + \gamma_{22} \gamma_{33} + \gamma_{33} \gamma_{11} - \gamma_{12} \gamma_{21} - \gamma_{13} \gamma_{31} - \gamma_{23} \gamma_{32}; \quad c_1 = \gamma_{33} + \gamma_{22} - \gamma_{12} - \gamma_{13} + \delta^2 \Lambda_1 d_1, \\ c_2 &= \gamma_{11} + \gamma_{33} - \gamma_{23} - \gamma_{21} + \delta^2 \Lambda_1 d_2; \quad c_3 = \gamma_{22} + \gamma_{11} - \gamma_{31} - \gamma_{32} + \delta^2 \Lambda_1 d_3; \quad c_4 = Pr_D b_7, \\ c_5 &= (1 + Pr_D \Lambda_2) d_4; \quad d_1 = \gamma_{23} (\gamma_{12} - \gamma_{32}) + \gamma_{13} (\gamma_{32} - \gamma_{22}) + \gamma_{33} (\gamma_{22} - \gamma_{12}), \\ d_2 &= \gamma_{31} (\gamma_{23} - \gamma_{13}) + \gamma_{21} (\gamma_{13} - \gamma_{33}) + \gamma_{11} (\gamma_{33} - \gamma_{23}), \\ d_3 &= \gamma_{12} (\gamma_{31} - \gamma_{21}) + \gamma_{32} (\gamma_{21} - \gamma_{11}) + \gamma_{22} (\gamma_{11} - \gamma_{31}), \\ d_4 &= \text{Det}(\gamma_{ij}) = \gamma_{11} (\gamma_{22} \gamma_{33} - \gamma_{23} \gamma_{32}) + \gamma_{12} (\gamma_{23} \gamma_{31} - \gamma_{21} \gamma_{33}) + \gamma_{13} (\gamma_{21} \gamma_{32} - \gamma_{22} \gamma_{31}). \end{aligned}$$

To study the linear instability of the system, the real part of σ is forced to zero and let $\sigma = i\omega$ in (16) and solve for R_1 , to obtain

$$R_1 = f_1(\omega^2, \alpha^2; \Lambda_1, \Lambda_2, R_2, R_3, \gamma_{ij}, Pr_D) + i\omega \delta^2 f_2(\omega^2, \alpha^2; \Lambda_1, \Lambda_2, R_2, R_3, \gamma_{ij}, Pr_D) \quad (i, j = 1, 2, 3), \quad (17)$$

where f_1 and f_2 are real valued functions of known parameters and they are not presented here because of lengthy expressions. Since R_1 is a real quantity, either $\omega = 0$ or $f_2 = 0$ in (17). The case $\omega = 0$ corresponds to stationary convection and the case $f_2 = 0$ ($\omega \neq 0$) corresponds to oscillatory convection. The transformations $-R_1 \rightarrow R_1$, $R_2 \rightarrow R_2$, and $R_3 \rightarrow R_3$ are introduced in the subsequent analysis.

3.1 | Stationary convection

The stationary convection begins at $R_1 = R_1^S$, where

$$R_1^S = \frac{1}{d_1} \left[d_2 R_2 + d_3 R_3 + d_4 \frac{\delta^4}{\alpha^2} \right]. \quad (18)$$

The stationary solute Rayleigh number R_1^S attains its minimum value at $\alpha = \pi$ and the minimum solute Rayleigh number for the stationary case is

$$R_{1c}^S = \frac{1}{d_1} [d_2 R_2 + d_3 R_3 + 4d_4 \pi^2]. \quad (19)$$

In the absence of cross-diffusion terms, the above-stated expression reduces to

$$R_{1c}^S = \frac{R_2}{\gamma_{22}} + \frac{R_3}{\gamma_{33}} + 4\pi^2, \quad (20)$$

and coincides with Raghunatha et al.³²

3.2 | Oscillatory convection

For this case, $\omega \neq 0$ but $f_2 = 0$ which gives a cubic dispersion relation in ω^2 of the form

$$m_1(\omega^2)^3 + m_2(\omega^2)^2 + m_3(\omega^2) + m_4 = 0, \quad (21)$$

where $m_1 - m_4$ are functions of α , Pr_D , Λ_1 , Λ_2 , R_2 , R_3 , γ_{ij} which are not presented as they are lengthy expressions. When $f_2 = 0$, (17) shows that oscillatory convection begins at $R_1 = R_1^o$, where

$$R_1^o = f_1(\omega^2, \alpha^2; \Lambda_1, \Lambda_2, R_2, R_3, \gamma_{ij}, Pr_D), \quad (22)$$

and ω^2 is given by (21). For any selected governing parameters, the minimum value of R_1^o with respect to α represented by R_{1c}^o , is determined using the procedure discussed by Raghunatha et al.³²

4 | RESULTS AND DISCUSSION

The impact of full cross-diffusion terms on the linear instability of triple-diffusive convection in an Oldroyd-B fluid-saturated porous layer is investigated. Depending on the types of interactions between the solute concentrations, the sign of cross-diffusion terms are considered to be either negative (counter-flux) or positive (co-flux). The numerical results are obtained for the lysozyme-sodium chloride-bovine serum albumin (BSA)-water system whose diffusivity matrix with and without cross-diffusion coefficients are, respectively, given by Vanag and Epstein⁶

$$D_{ij} = \begin{bmatrix} 1.26 & -0.55 & -104 \\ -0.42 & 1.32 & 60 \\ -0.00013 & 0.00004 & 0.07 \end{bmatrix} \times 10^{-5} \text{cm}^2 \text{s}^{-1}, \quad (23)$$

and

$$D'_{ij} = \begin{bmatrix} 1.26 & 0 & 0 \\ 0 & 1.04762 & 0 \\ 0 & 0 & 0.05556 \end{bmatrix} \times 10^{-5} \text{cm}^2 \text{s}^{-1}. \quad (24)$$

The Darcy-Prandtl number assumes the value, $Pr_D = 714$. To assess the sensitivity of the off-diagonal elements on the instability of the system, the diffusivity matrix is parameterized as follows:

$$D_{ij} = \begin{bmatrix} D'_{11} + \chi(D_{11} - D'_{11}) & \chi D_{12} & \chi D_{13} \\ \chi D_{21} & D'_{22} + \chi(D_{22} - D'_{22}) & \chi D_{23} \\ \chi D_{31} & \chi D_{32} & D'_{33} + \chi(D_{33} - D'_{33}) \end{bmatrix}, \quad (25)$$

where χ is the cross-diffusion sensitivity parameter. The cases $\chi \neq 0$ and $\chi = 0$ correspond to with and without cross-diffusion effects, respectively.

The critical solute Darcy-Rayleigh numbers (R_{1c}) computed for different values of R_2 and R_3 are tabulated in Table 1 for Maxwell and Oldroyd-B fluids. The superscript # denotes the results obtained in the presence of cross-diffusion terms. For the considered governing parameters, it is seen that R_{1c} is either positive/negative indicating stabilizing/destabilizing influence on the viscoelastic fluid-saturated

TABLE 1 Comparisons between critical frequency and critical Rayleigh numbers for viscoelastic fluids for diffusivity data given by matrix (25)

Oscillatory onset									
R_2	R_3	Maxwell fluid ($\Lambda_1 = 2, \Lambda_2 = 0$)				Oldroyd-B fluid ($\Lambda_1 = 2, \Lambda_2 = 1$)			
		ω_c	$\omega_c^\#$	R_{1c}	$R_{1c}^\#$	ω_c	$\omega_c^\#$	R_{1c}	$R_{1c}^\#$
−10	1	28.484	154.910	−11.0484	−71.6388	4.94044	30.7407	10.1737	3.14062
−10	2	36.923	216.63	−10.6761	−133.533	6.41678	49.3466	11.281	1.67753
−10	5	55.744	340.253	−9.94416	−319.231	9.63878	201.866	14.4203	−13.5187
−10	10	78.115	480.134	−9.14421	−628.731	13.5155	422.491	19.4725	−108.852
−10	15	95.509	587.612	−8.55160	−938.233	16.6139	577.233	24.4210	−248.679

porous layer. The cross-diffusion can either stabilize or destabilize the porous layer in the case of Oldroyd-B fluids, while the system turns out to be only stabilizing in its absence. Irrespective of the cross-diffusion effect, the porous layer is always destabilizing in the case of Maxwell fluids. Furthermore, increase in the value of R_3 is to increase the frequency of oscillations in the absence and presence of cross-diffusion for Oldroyd-B and Maxwell fluids.

Figure 1 shows the variation of critical solute Darcy-Rayleigh number as a function of Λ_2 for various values of relaxation parameter Λ_1 in the presence and absence of cross-diffusion terms for $R_2 = 1$ and $R_3 = -1$. It is seen that the critical oscillatory solute Darcy-Rayleigh number increases with Λ_2 for each value of Λ_1 . Some threshold value exceeds for Λ_2 , then the instability ceases to be oscillatory and becomes steady and this value increases with increasing Λ_1 . The presence of cross-diffusion decreases the values of R_{1c} compared with its absence and to hasten the onset of both stationary and oscillatory convection. The consequence of increasing Λ_1 and Λ_2 is to hasten and delay the onset of oscillatory convection, respectively.

A systematic study on the neutral stability curves reveals some novel results on the nature of instability not found in double-diffusive viscoelastic fluid systems in the presence of cross-diffusion terms. Figure 2A-D shows the effect of cross-diffusion sensitivity parameter χ on the evolution of neutral stability curves for $\Lambda_1 = 10$, $\Lambda_2 = 5$, $R_2 = 200$, $R_3 = -0.1$ and for the diffusivity matrix given by (25). Figure 2A shows the results for $\chi = 0$ (without cross-diffusion terms) and note that the oscillatory neutral curve lies below the stationary neutral curve and a single critical solute Darcy-Rayleigh number is sufficient to dwell upon fully the instability of the system. However, an altogether different scenario is evinced at $\chi = 0.1$ (Figure 2B). This figure demonstrates the formation of a closed oscillatory neutral curve completely detaching from the stationary neutral curve owing to the existence of two frequencies and hence two oscillatory solute Darcy-Rayleigh numbers for a fixed wavenumber. It is further seen that the closed oscillatory neutral curve has two maxima at different values of solute Darcy-Rayleigh number. That is, the quasiperiodic bifurcation to convection from the motionless state is found to be not possible in the present case. The closed disconnected oscillatory neutral curve goes on diminishing with increasing $\chi = 0.2$ (Figure 2C) and disappears when $\chi = 0.22$ (Figure 2D) leaving only the stationary neutral curve. The closed disconnected oscillatory neutral curve has important implications on the linear instability criteria. That is, three critical values of R_1 are needed to specify the linear instability criteria (Figure 2C) and may be stated as follows. For $R_1 < R_{11}$ and $R_{12} < R_1 < R_{13}$, the viscoelastic fluid-saturated porous layer is linearly stable. For

FIGURE 1 Variation of R_{1c} with Λ_2 for different values of Λ_1 when $R_2 = 1$, $R_3 = -1$, and for the diffusivity matrix given by (25)

FIGURE 2 Evolution of neutral stability curves for (A) $\chi = 0$, (B) $\chi = 0.1$, (C) $\chi = 0.2$, and (D) $\chi = 0.22$ with $\Lambda_1 = 10$, $\Lambda_2 = 5$, $R_2 = 200$, $R_3 = -0.1$, and for the diffusivity matrix of (25)

$R_{11} < R_1 < R_{12}$ and $R_1 > R_{13}$, the porous layer is unstable. It is thus evident that the presence of cross-diffusion terms has a significant influence on the nature of instability.

The evolution of neutral stability curves are displayed for another set of governing parameters in Figure 3A–D for different values of χ . The results presented here are for $\Lambda_1 = 10$, $\Lambda_2 = 9$, $R_2 = 189$, $R_3 = -0.3$, and for the diffusivity matrix given by (25). Figure 3A illustrates that for $\chi = 0$, the oscillatory neutral curve attached to the stationary neutral curve at two bifurcation points and the pinching of the oscillatory part on either side has given rise to two oscillatory neutral solutions for some wave numbers. The bifurcation points move closer and forms a closed convex oscillatory neutral curve with slight increase in the value of $\chi (= 0.007496)$, as illustrated in Figure 3B. These figures still indicate the sufficiency of a single critical solute Darcy-Rayleigh number to stipulate the linear instability criteria. The closed convex oscillatory neutral curve moves well below the minimum of the stationary neutral curve for a very small increased value of $\chi = 0.00751$, as shown in Figure 3C, representing the necessity of three critical values of R_1 to

FIGURE 3 Evolution of neutral stability curves for (A) $\chi = 0$, (B) $\chi = 0.007496$, (C) $\chi = 0.00751$, and (D) $\chi = 0.00752$ with $\Lambda_1 = 10$, $\Lambda_2 = 9$, $R_2 = 189$, $R_3 = -0.3$, and for the diffusivity matrix of (25)

FIGURE 4 Stability boundary for $\Lambda_1 = 10$, $\Lambda_2 = 9$, $R_2 = 189$, $R_3 = -0.3$, and for the diffusivity matrix given by (25)

completely identify the linear instability criteria. Finally, the oscillatory closed neutral curve vanishes leaving only the stationary neutral curve when χ takes the value 0.00752. These figures clearly portray the significance of cross-diffusion terms in proper understanding of convective instability in multicomponent fluid systems. In other words, even minute variations in the cross-diffusion sensitivity parameter completely amend the nature of instability of the system.

Figure 4 illustrates the stability boundaries for the same parametric values considered in Figure 3 and they can be viewed in three regions depending on the values of χ . To the left of the point of G ($\chi < 0.00749597555$), oscillatory convection first begins at a lower value of R_1 than does the stationary convection and there is only one value of R_{1c} . To the right of the point H, ($\chi > 0.00751164799$), stationary instability occurs and oscillatory instability does not happen and then there is one value of R_{1c} . But the interesting multivalued nature of the stability boundary portion lies in the range $0.00749597555 < \chi < 0.00751164799$. In this finite range of χ , three R_{1c} values are found to be necessary to recognize the linear instability criteria.

FIGURE 5 Effect of cross-diffusion terms on neutral stability curves for the diffusivity data given by (25), $R_2 = 200$, and $R_3 = -0.1$: A, Oldroyd-B fluid: $\Lambda_1 = 10$, $\Lambda_2 = 5$; B, Maxwell fluid: $\Lambda_1 = 10$, $\Lambda_2 = 0$; C, Newtonian fluid: $\Lambda_1 = \Lambda_2 = 0$

The similarity and differences between Oldroyd-B, Maxwell, and Newtonian models with cross-diffusion and without cross-diffusion effects are displayed in Figure 5A-C, respectively. From these figures, we note that for an Oldroyd-B fluid case, three critical solute Darcy-Rayleigh numbers are required to fully specify the linear instability criteria in the presence of cross-diffusion terms. In the absence of cross-diffusion terms, however, only one critical solute Darcy-Rayleigh number is sufficient to specify the linear instability criteria. Thus, the presence of cross-diffusion alters the instability of the system significantly in the case of Oldroyd-B fluids (Figure 5A). In the case of Maxwell fluids (Figure 5B), oscillatory convection is found to be a preferred mode of instability but a single critical solute Darcy-Rayleigh number is sufficient to specify the instability of the system with or without cross-diffusion effects. The circumstances observed for Newtonian fluids (Figure 5C) is, nonetheless, different from those of Oldroyd-B and Maxwell fluids and note that only stationary convection is possible irrespective of the cross-diffusion effects.

5 | CONCLUSIONS

The intricacies of cross-diffusion on the linear instability of triply diffusive convection in a viscoelastic fluid-saturated porous layer are examined and closed-form solutions are obtained by performing the normal mode analysis. The viscoelastic fluid motion in porous media is modeled by a modified Darcy-Oldroyd-B equation that encompasses the Maxwell model and the Newtonian fluid model. In analyzing the instability analysis, all the nine elements in the diffusivity matrix are considered. The numerical calculations of critical solute Darcy-Rayleigh number revealed inconsistencies in the nature of instability criteria between the absence and presence of cross-diffusion. The cross-diffusion is either stabilizing or destabilizing depending on the magnitude of solute Darcy-Rayleigh numbers R_2 and R_3 , and also on the type of viscoelastic fluid (Maxwell/Oldroyd-B). The viscoelastic parameters exhibit an opposite kind of behavior on the onset and the range of retardation parameter within which the oscillatory convection is favored increases with increasing relaxation parameter. The instability characteristics of the system analyzed under the same values of governing parameters for an Oldroyd-B, Maxwell, and Newtonian fluids are found to be qualitatively different. Some novel consequences not observed in a viscoelastic double-diffusive fluid-saturated porous medium with cross-diffusion effects have been identified. First, the requirement of three critical solute Darcy-Rayleigh number to specify the linear instability criteria instead of the traditional single value and second, the onset of instability does not occur simultaneously at the same critical solute Darcy-Rayleigh number at different wavenumbers, that is, heart-shaped oscillatory neutral curve with twin maxima is not found to occur instead closed convex oscillatory neutral curve with single maximum exists. More importantly, it is found that even small variations in the cross-diffusion elements of the diffusivity matrix amount to qualitative differences in the stability criteria of triply cross-diffusive Oldroyd-B fluid-saturated porous layer.

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REFERENCES

1. Celia MA, Kindred JS, Herrera I. Contaminant transport and biodegradation: 1. A numerical model for reactive transport in porous media. *Water Resour Res.* 1989;25:1141-1148.
2. Chen B, Cunningham A, Ewing R, Peralta R, Visser E. Two-dimensional modelling of micro scale transport and biotransformation in porous media. *Numer Methods Partial Differ Equations.* 1994;10:65-83.
3. Onsager L, Fuoss RM. Irreversible processes in electrolytes. Diffusion, conductance and viscous flow in arbitrary mixtures of strong electrolytes. *J Phys Chem.* 1932;36:2689-2778.
4. Baldwin RL, Dunlop PJ, Gosting LJ. Interacting flows in liquid diffusion: equations for evaluation of the diffusion coefficients from moments of the refractive index gradient curves. *J Am Chem Soc.* 1955;77:5235-5238.
5. Mimura M, Kawasaki K. Spatial segregation in competitive interaction-diffusion equations. *J Math Biol.* 1980;9:49-64.
6. Vanag VK, Epstein IR. Cross-diffusion and pattern formation in reaction-diffusion systems. *Phys Chem Chem Phys.* 2009;11:897-912.
7. Annunziata O, Vergara A, Paduano L, Sartorio R, Miller DG, Albright JG. Quaternary diffusion coefficients in a protein-polymer-salt-water system determined by Rayleigh interferometry. *J Phys Chem B.* 2009;113:13446-13453.
8. Griffiths RW. The influence of a third diffusing component upon the onset of convection. *J Fluid Mech.* 1979;92:659-670.
9. Rudraiah N, Vortmeyer D. Influence of permeability and of a third diffusing component upon the onset of convection in a porous medium. *Int J Heat Mass Transfer.* 1982;25:457-464.
10. Poulikakos D. Effect of a third diffusing component on the onset of convection in a horizontal layer. *Phys Fluids.* 1985;28:3172-3174.
11. Tracey J. Multi-component convection-diffusion in a porous medium. *Continuum Mech Thermodyn.* 1996;8:361-381.
12. Pearlstein AJ, Harris RM, Terrones G. The onset of convective instability in a triply diffusive of fluid layer. *J Fluid Mech.* 1989;202:443-465.
13. Rionero S. Long-time behaviour of multi-component fluid mixtures in porous media. *Int J Eng Sci.* 2010;48:1519-1533.
14. Rionero S. Triple diffusive convection in porous media. *Acta Mech.* 2013;224:447-458.
15. Shenoy AV. Non-Newtonian fluid heat transfer in porous media. *Adv Heat Transf.* 1994;24:101-190.
16. Khuzhavorov B, Auriault JL, Rover P. Derivation of macroscopic filtration law for transient linear viscoelastic fluid flow in porous media. *Int J Eng Sci.* 2000;38:487-504.
17. Khaled AR, Vafai K. The role of porous media in modeling flow and heat transfer in biological tissues. *Int J Heat Mass Transfer.* 2003;46:7989-5003.
18. Tan W, Masuoka T. Stokes' first problem for an Oldroyd-B fluid in a porous half space. *Phys Fluids.* 2005;17:023101-023108.
19. Rudraiah N, Kaloni PN, Radhadevi PV. Oscillatory convection in a viscoelastic fluid through a porous layer heated from below. *Rheol Acta.* 1989;28:48-53.
20. Kim MC, Lee SB, Kim S, Chung BJ. Thermal instability of viscoelastic fluids in porous media. *Int J Heat Mass Transfer.* 2003;46:5065-5072.
21. Yoon DY, Kim MC, Choi CK. The onset of oscillatory convection in a horizontal porous layer saturated with viscoelastic liquid. *Transp Porous Media.* 2004;55:275-284.
22. Bertola V, Cafaro E. Thermal instability of viscoelastic fluids in horizontal porous layers as initial problem. *Int J Heat Mass Transfer.* 2006;49:4003-4012.
23. Tan W, Masuoka T. Stability analysis of a Maxwell fluid in a porous medium heated from below. *Phys Lett A.* 2007;360:454-460.
24. Shivakumara IS, Sureshkumar S. Convective instabilities in a viscoelastic-fluid-saturated porous medium with through flow. *J Geophys Eng.* 2007;4:104-115.
25. Raghunatha KR, Shivakumara IS, Sowbhagya. Stability of buoyancy-driven convection in an Oldroyd-B fluid-saturated anisotropic porous layer. *Appl Math Mech-Engl Ed.* 2018;39(5):653-666.
26. Malashetty MS, Tan W, Swamy M. The onset of double diffusive convection in a binary viscoelastic fluid-saturated anisotropic porous layer. *Phys Fluids.* 2009;21:084101-084111.

27. Wang S, Tan W. Stability analysis of double-diffusive convection of Maxwell fluid in a porous medium heated from below. *Phys Lett A*. 2008;372:3046-3050.
28. Wang S, Tan W. Stability analysis of solet-driven double-diffusive convection of Maxwell fluid in a porous medium. *Int J Heat Fluid Flow*. 2011;32:88-94.
29. Shivakumara IS, Sureshkumar S. Effects of throughflow and quadratic drag on the stability of a doubly diffusive Oldroyd-B fluid-saturated porous layer. *J Geophys Eng*. 2008;5:268-280.
30. Altawallbeh AA, Hashim I, Bhadauria BS. On the linear stability analysis of double diffusive convection in a viscoelastic fluid saturated porous layer with cross diffusion effects and internal heat source. In AIP Conference Proceedings. 2017;1830:020008-020016.
31. Zhao M, Wang S, Zhang Q. Onset of triply diffusive convection in a Maxwell fluid-saturated porous layer. *Appl Math Model*. 2014;38:2345-2352.
32. Raghunatha KR, Shivakumara IS, Shankar BM. Weakly nonlinear stability analysis of triple diffusive convection in a Maxwell fluid saturated porous layer. *Appl Math Mech*. 39, 2018:153-168.
33. Kalita D, Karmakar PK. Nonlinear dynamics of gravitational instability in complex viscoelastic astrofluids. *AIP Adv*. 2018;8:085207.
34. Karmakar PK, Kalita D. Dynamics of gravitational instability excitation in viscoelastic polytropic fluids. *Astrophys Space Sci*. 2018;363:239-240.
35. Malashetty MS, Biradar BS. Linear and nonlinear double-diffusive convection in a fluid-saturated porous layer with cross-diffusion effects. *Transp Porous Media*. 2012;9:649-675.
36. Hutter K, Straughan B. Models for convection in thawing porous media in support for the subsea permafrost equations. *J Geophys Res Solid Earth*. 1999;104:29249-29260.
37. Bahloul A, Boutana N, Vasseur P. Double-diffusive and Soret-induced convection in a shallow horizontal porous layer. *J Fluid Mech*. 2003;491:325-352.
38. Mansour A, Amahmid A, Hasnaoui M, Bourich M. Multiplicity of solutions induced by thermosolutal convection in a square porous cavity heated from below and submitted to horizontal concentration gradient in the presence of Soret effect. *Numer Heat Transfer A, Appl*. 2006;49:69-94.
39. Roy K, Murthy PVS. Linear stability of the double-diffusive convection in a horizontal porous layer with open top: Soret and viscous dissipation effects. *Transp Porous Med*. 2018;122:693-712.
40. Kim MC, Song KH. Effect of cross-diffusion on the gravitational instability in a ternary mixture: asymptotic and linear analyses. *Chem Eng Sci*. 2018;191:191-199.
41. Kim MC, Song KH. Theoretical and numerical analyses of the effect of cross-diffusion on the gravitational instability in ternary mixtures. *Int J Heat Mass Transfer*. 2019;143:118511-118529.
42. Awad FG, Sibanda P, Motsa SS. On the linear stability analysis of a Maxwell fluid with double-diffusive convection. *Appl Math Model*. 2010;34:3509-3517.
43. Malashetty MS, Biradar BS. The onset of double diffusive convection in a binary Maxwell fluid saturated porous layer with cross-diffusion effects. *Phys Fluids*. 2011;23:064109-064122.
44. Terrones G. Cross diffusion effects on the stability criteria in a triply diffusive system. *Phys Fluids A*. 1993;5: 2172-2182.
45. Raghunatha KR, Shivakumara IS, Swamy MS. Effect of cross-diffusion on the stability of a triple-diffusive Oldroyd-B fluid layer. *Z Aangew Math Phys*. 2019;70:100. <https://doi.org/10.1007/s00033-019-1143-x>
46. Alishaev MG, Mirzadzandeh AK. For the calculation of delay phenomenon in filtration theory. *Izv Vyssh Uchebn Zaved Neft Gaz*. 1975;6:71-78.
47. Nield DA, Bejan A. *Convection in Porous Media*. 5th ed. New York, NY: Springer-Verlag; 2017.

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