Study of Rayleigh–Bénard Convection of a Newtonian Nanoliquid in a High Porosity Medium Using Local Thermal Non-equilibrium Model

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Abstract

In the paper we make linear and non-linear stability analyses of Rayleigh–Bénard convection in a Newtonian, nanoliquid-saturated porous medium using local thermal non-equilibrium model (LTNE). The LTNE assumption results in advanced onset of convection and increase in heat transport when compared to that of local thermal equilibrium assumption. Free–free and rigid–rigid, isothermal boundaries are considered for investigation. The Galerkin method is used to obtain the critical eigen value. The influence of inter-phase heat transfer coefficient, ratio of thermal conductivities, Brinkman number, porous parameter on the onset of convection as well as on heat transport has been presented graphically and discussed in detail. The effect of increasing the value of porosity modified thermal conductivities advances the onset of convection and enhances the amount of heat transport whereas the remaining parameters have an opposing influence on both onset of convection as well as heat transport.

Keywords Rayleigh–Bénard convection · Nanoliquid · Porous medium · LTNE · Free–free · Rigid–rigid · Linear · Non-linear

List of Symbols

A, B, C, D	Amplitudes of convection
C_p	Specific heat at constant pressure
d	Channel depth
g	Acceleration due to gravity
h	Inter-phase heat transfer coefficient
H	Dimensionless inter-phase heat transfer coefficient
k	Wavenumber in the x direction

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K	Permeability
p	Pressure
Pr	Prandtl number
$\vec{q}$	Filtration velocity or Darcy velocity
t	Time
T	Temperature
V	Dimensional velocity in the y direction
x, z	Cartesian coordinate
X, Z	Dimensionless coordinates

Greek Symbols

α_{nl}	Thermal diffusivity of the nanoliquid
α_{bl}	Thermal diffusivity of the base liquid
γ	Porosity-modified ratio of thermal conductivities
β	Thermal expansion coefficient
k	Thermal conductivity
Λ	Ratio of viscosities or Brinkman number
μ	Viscosity of the nanoliquid
μ'	Viscosity of the nanoliquid in saturated porous medium
ϕ	Porosity ($0 < \phi < 1$)
Ψ	Dimensionless stream function
ψ	Dimensional stream function
ρ	Density
σ^2	Inverse Darcy number or porous parameter
τ	Dimensionless time
Θ	Dimensionless temperature

Subscripts or Superscripts

0	Reference value
b	Basic state
c	Critical
bl	Base liquid
nl	Nano liquid
LTE	Local thermal equilibrium
$LTNE$	Local thermal non-equilibrium
s	Solid

Introduction

Thermal convection in fluid-saturated porous media is of interest with respect to many engineering and environmental applications, such as ice production, food cooling, crystal growth and others. The studies are mainly based on the assumption that the fluid and porous medium phases are everywhere in local thermal equilibrium (LTE) where the temperature gradient at any location between the two phases is assumed to be negligible. However, in many practical applications involving large temperature differences between the fluid and solid phases the assumption of LTE is inadequate. In such circumstances, it is pertinent to take account of the local thermal non-equilibrium (LTNE) effects by considering a two-field model for the energy equation, one each for the fluid and solid phases. The studies on convective flows in

a porous medium considering LTNE effects is well documented in the book by Nield and Bejan [11].

The problem in a non-rotating porous medium using LTNE has been investigated by many authors including Banu and Rees [3], Postelnicu and Rees [15], Rees and Pop [16], Straughan [20], Postelnicu [14], Nield and Kuznetsov [12], Barletta et al. [4], Nield et al. [13], Celli et al. [5], Siddheshwar and Siddabasappa [19] and many more. Asymptotic analyses of critical wave number, Rayleigh number and Nusselt number in the case of free–free and rigid–rigid, isothermal boundaries using LTNE is carried out by Siddheshwar and Siddabasappa [19].

Convection in a nanoliquid-saturated porous medium is also important as combination of both nanoliquids and porous medium can be used to regulate the residence time of heat in the system. Few papers have reported on the above problem [1,2,18,19]. In the present problem, we concentrate on the unconsidered aspects of the Brinkman–Bénard convection. Most of the works in literature consider the unrealistic free–free boundary conditions. The current study makes use of more realistic rigid–rigid boundaries. The limiting behaviour of H , γ and γH to obtain the results of LTE in case of rigid–rigid boundaries is clearly showcased in the present study through asymptotic analysis which is not seen in earlier works. This makes the problem more interesting and challenging than when considered under LTE assumption. There seems to be no work on Rayleigh–Bénard convection in a nanoliquid-saturated porous layer using LTNE model for free–free and rigid–rigid, isothermal boundaries

The objective of the paper is to investigate the following aspects in the current problem:

1. Studying the onset of Rayleigh–Bénard convection and amount of heat transport in a nanoliquid-saturated porous medium using free–free and rigid–rigid, isothermal boundaries.
2. Obtaining the results on Rayleigh–Bénard convection of Newtonian base liquids as a limiting case of the current problem.
3. Obtaining the results on Rayleigh–Bénard convection of Newtonian base liquids and nanoliquids using LTE assumption as a limiting case.
4. Regulating the residence time of heat in the system using nanoparticles and porous matrix.
5. Making predictions on the onset of convection as well as heat transport by varying various non-dimensional parameters.
6. Performing the asymptotic analysis of critical wave number, Rayleigh number and Nusselt number for small and large H , large γ and large γH .

Mathematical Formulation

The schematic of the flow configuration is as shown in Fig. 1.

The governing equations for the problem are:

$$\nabla \cdot \vec{q} = 0, \quad (1)$$

$$\frac{\rho_{nl}}{\phi} \left(\frac{\partial \vec{q}}{\partial t} + \frac{1}{\phi} (\vec{q} \cdot \nabla) \vec{q} \right) = -\nabla p + \mu' \nabla^2 \vec{q} - \frac{\mu}{K} \vec{q} + \rho_{nl} \vec{g}, \quad (2)$$

$$(\rho C_p)_{nl} \left(\phi \frac{\partial T_{nl}}{\partial t} + (\vec{q} \cdot \nabla) T_{nl} \right) = \phi k_{nl} \nabla^2 T_{nl} + h(T_s - T_{nl}), \quad (3)$$

$$(1 - \phi)(\rho C_p)_s \frac{\partial T_s}{\partial t} = (1 - \phi)k_s \nabla^2 T_s - h(T_s - T_{nl}), \quad (4)$$

$$\rho(T) = \rho(T_0)[1 - \beta_{nl}(T_{nl} - T_0)]. \quad (5)$$

Fig. 1 Schematic of the physical configuration

Table 1 Thermophysical properties of 30% glass fiber reinforced polycarbonate porous material at 300 K [8]

Density (kg m^{-3})	Specific heat (J/kg K)	Thermal conduc- tivity (W/m K)	Thermal diffusivity ($\text{m}^2 \text{s}^{-1} \times 10^7$)	Porosity (ϕ)
$(\rho)_s = 1430$	$(C_p)_s = 1130$	$k_s = 0.24$	$(\alpha)_s = 1.4852$	0.88

Table 2 Thermo-physical properties of ethylene glycol–copper saturated porous medium at 300 K for volume fraction, $\chi = 0.06$ and $\phi = 0.88$

Quantity	Ethylene glycol–copper saturated porous medium
Density (ρ) (kg/m^3)	1565.09
Specific heat (C_p) (J/kg K)	1662.34
Thermal conductivity (k) (W/m K)	0.29294
Thermal expansion coefficient (β) ($\text{K}^{-1} \times 10^5$)	39.17174
Dynamic coefficient of viscosity (μ') (kg/m s)	0.02522673
Thermal diffusivity (α) ($\text{m}^2 \text{s}^{-1} \times 10^7$)	1.12545
(ρC_p) ($\text{J/m}^3 \text{K} \times 10^{-6}$)	2.60172
$(\rho\beta)$ ($\text{kg/m}^3 \text{K}$)	0.613073

We assume boundary conditions on $\vec{q}$ and T in the basic static state to be:

$$\begin{aligned}\vec{q} &= 0, T = T_0 + \Delta T \text{ at } z = -\frac{d}{2}, \\ \vec{q} &= 0, T = T_0 \text{ at } z = \frac{d}{2}.\end{aligned}$$

Further, the boundaries are assumed to be free–free or rigid–rigid. Taking the velocity, temperature, density and pressure in the quiescent basic state as follows:

$$\left. \begin{aligned}\vec{q} &= \vec{q}_b = (0, 0), \\ T_{nl}(z) &= T_{nl_b}(z), \quad T_s(z) = T_{s_b}(z), \\ \rho(z) &= \rho_b(z), \quad p(z) = p_b(z),\end{aligned} \right\}, \quad (6)$$

Fig. 2 Variation of Nu_w with Ra_{nl} of free–free and rigid–rigid boundaries, for $\Lambda = 1.2$, $\sigma^2 = 5$ and $\gamma = 10$, for small values of H

we obtain the quiescent state solution for the temperature in the form:

$$T_{nl_b}(z) = T_0 + \Delta T \left(\frac{1}{2} - \frac{z}{d} \right), \quad (7)$$

$$T_{sb}(z) = T_0 + \Delta T \left(\frac{1}{2} - \frac{z}{d} \right). \quad (8)$$

The other basic state quantities are not required explicitly in the analysis.

We now superimpose perturbation on the quiescent basic state and so we write:

$$\left. \begin{aligned} \vec{q} &= \vec{q}_b + \vec{q}'(x, z, t), \\ T_{nl} &= T_{nl_b} + T'(x, z, t), \\ T_s &= T_{sb} + T'(x, z, t), \\ \rho &= \rho_b + \rho'(x, z, t), \\ p &= p_b + p'(x, z, t), \end{aligned} \right\}, \quad (9)$$

where the primes indicate a perturbed quantity. The pressure p is eliminated between the x - and y -components of Eq. (2) and then the stream function $\psi(x, z, t)$ is introduced in the form

Fig. 3 Variation of Nu_w with Ra_{nl} of free–free and rigid–rigid boundaries, for $\Lambda = 1.2$, $\sigma^2 = 5$ and $\gamma = 10$, for large values of H

$$u = \frac{\partial \psi}{\partial z} \text{ and } w = -\frac{\partial \psi}{\partial x}.$$

The resulting equations are rendered dimensionless using the following scaling:

$$(X, Z) = \left(\frac{x}{d}, \frac{z}{d} \right), \quad V = \left(\frac{(\rho C_p)_{bl} d}{\phi k_{bl}} \right) v, \quad \Psi = \left(\frac{(\rho C_p)_{bl}}{\phi k_{bl}} \right) \psi, \quad \Theta_{nl} = \frac{T_{nl}}{\Delta T},$$

$$\Theta_s = \frac{T_s}{\Delta T}, \quad \tau = \frac{k_{bl}}{(\rho C_p)_{bl} d^2} t. \quad (10)$$

Using the scaling (10), the dimensionless form of the governing equations are:

$$\frac{1}{Pr_{nl}} \frac{\partial}{\partial \tau} (\nabla^2 \Psi) = a_1 \Lambda \nabla^4 \Psi - a_1 \sigma^2 \nabla^2 \Psi - a_1^2 Ra_{nl} \frac{\partial \Theta_{nl}}{\partial X}, \quad (11)$$

$$\frac{\partial \Theta_{nl}}{\partial \tau} = -\frac{\partial \Psi}{\partial X} + a_1 \nabla^2 \Theta_{nl} + a_1 H(\Theta_s - \Theta_{nl}) + \frac{\partial (\Psi, \Theta_{nl})}{\partial (X, Z)}, \quad (12)$$

$$\Gamma \frac{\partial \Theta_s}{\partial \tau} = a_1 \nabla^2 \Theta_s + a_1 \gamma H(\Theta_{nl} - \Theta_s), \quad (13)$$

Fig. 4 Variation of Nu_w with Ra_{nl} of free–free and rigid–rigid boundaries, for $\Lambda = 1.2$, $\sigma^2 = 5$ and $H = 10$, for large values of γ

where

$$Pr_{nl} = \frac{\phi \mu_{nl} (\rho C_p)_{nl}}{\rho_{nl} k_{nl}} \text{ (Prandtl number),}$$

$$a_1 = \frac{\alpha_{nl}}{\alpha_{bl}} \text{ (ratio of thermal diffusivities),}$$

$$Ra_{nl} = \frac{(\rho \beta)_{nl} (\rho C_p)_{nl} g d^3 \Delta T}{\phi k_{nl} \mu_{nl}} \text{ (Rayleigh number),}$$

$$\Lambda = \frac{\mu'_{nl}}{\mu_{nl}} \text{ (Brinkman number), } \sigma^2 = \frac{d^2}{K} \text{ (porous parameter),}$$

$$\Gamma = \frac{k_{nl} (\rho C_p)_s}{k_s (\rho C_p)_{nl}} = F a_1, \quad F = \frac{\alpha_{bl}}{\alpha_s}$$

$$H = \frac{h d^2}{\phi k_{nl}} \text{ (dimensionless inter-phase heat transfer coefficient),}$$

$$\gamma = \frac{\phi k_{nl}}{(1 - \phi) k_s} \text{ (porosity modified ratio of thermal conductivities).}$$

Fig. 5 Variation of Nu_w with Ra_{nl} of free–free and rigid–rigid boundaries, for $\Lambda = 1.2$, $\sigma^2 = 5$ and $H = 10$, for large values of γH

We now carry out a linear stability analysis and study the onset of convection.

Linear Stability Analysis for Marginal Stationary Convection for Free–Free/Rigid–Rigid, Isothermal Boundaries

In order to make a linear stability analysis, we consider the linear and steady-state version of Eqs. (11)–(13).

Free–Free, Isothermal Boundaries

The boundary conditions suitable for free–free isothermal boundaries are:

$$\Psi = \frac{\partial^2 \Psi}{\partial Z^2} = \Theta_{nl} = \Theta_s = 0 \quad \text{at } Z = \pm \frac{1}{2}. \quad (14)$$

Table 3 Comparison of exact and asymptotic solutions for small values of H and fixed values of $\gamma = 0.01$, $\Lambda = 1.2$ and $\sigma^2 = 5$ in case of free–free boundaries

$\log_{10} H$	$v_c(E)$	$v_c(A)$	$Ra_{nlc}(E)$	$Ra_{nlc}(A)$
−3.5	2.61353	2.61353	1839.05	1839.05
−3	2.61357	2.61357	1839.13	1839.13
−2.5	2.61368	2.61368	1839.36	1839.36
−2	2.61402	2.61402	1840.12	1840.12
−1.5	2.61511	2.61511	1842.5	1842.5
−1	2.61855	2.61855	1850.02	1850.02
−0.5	2.62927	2.62927	1873.75	1873.75
0	2.66175	2.6616	1948.25	1948.25
0.5	2.75239	2.7483	2179.21	2179.22

E-exact value, A-asymptotic value

Table 4 Comparison of exact and asymptotic solutions for large values of H and fixed values of $\gamma = 10$, $\Lambda = 1.2$ and $\sigma^2 = 5$ in case of free–free boundaries

$\log_{10} H$	$v_c(E)$	$v_c(A)$	$Ra_{nlc}(E)$	$Ra_{nlc}(A)$
1	2.62313	2.6224	1996.57	1997.31
1.5	2.6172	2.61718	2013.69	2013.71
2	2.61476	2.61476	2019.9	2019.9
2.5	2.61392	2.61392	2021.95	2021.95
3	2.61364	2.61364	2022.61	2022.61
3.5	2.61356	2.61356	2022.82	2022.82
4	2.61353	2.61353	2022.89	2022.89
4.5	2.61352	2.61352	2022.91	2022.91
5	2.61352	2.61352	2022.91	2022.91
5.5	2.61352	2.61352	2022.92	2022.92
6	2.61352	2.61352	2022.92	2022.92

E-exact value, A-asymptotic value

Table 5 Comparison of exact and asymptotic solutions for large values of γ and fixed values of $H = 5$, $\Lambda = 1.2$ and $\sigma^2 = 5$ in case of free–free boundaries

$\log_{10} \gamma$	$v_c(E)$	$v_c(A)$	$Ra_{nlc}(E)$	$Ra_{nlc}(A)$
1	2.62843	2.64175	1976.79	1961.24
1.5	2.61577	2.61634	1891.61	1891.03
2	2.61378	2.6138	1856.81	1856.79
2.5	2.61354	2.61354	1844.77	1844.77
3	2.61352	2.61352	1840.85	1840.85
3.5	2.61352	2.61352	1839.6	1839.6
4	2.61352	2.61352	1839.2	1839.2
4.5	2.61352	2.61352	1839.07	1839.07
5	2.61352	2.61352	1839.03	1839.03
5.5	2.61352	2.61352	1839.02	1839.02
6	2.61352	2.61352	1839.02	1839.02

E-exact value, A-asymptotic value

Table 6 Comparison of exact and asymptotic solutions for large values of γH and fixed values of $H = 10$, $\Lambda = 1.2$ and $\sigma^2 = 5$ in case of free–free boundaries

$\log_{10} \gamma H$	$v_c(E)$	$v_c(A)$	$Ra_{nlc}(E)$	$Ra_{nlc}(A)$
2	2.62313	2.62763	1996.57	1992.14
2.5	2.61475	2.61493	1894.25	1894.1
3	2.61365	2.61366	1857.1	1857.1
3.5	2.61353	2.61353	1844.8	1844.8
4	2.61352	2.61352	1840.85	1840.85
4.5	2.61352	2.61352	1839.6	1839.6
5	2.61352	2.61352	1839.2	1839.2
5.5	2.61352	2.61352	1839.07	1839.07
6	2.61352	2.61352	1839.03	1839.03
6.5	2.61352	2.61352	1839.02	1839.02
7	2.61352	2.61352	1839.02	1839.02
7.5	2.61352	2.61352	1839.02	1839.02
8	2.61352	2.61352	1839.02	1839.02

E-exact value, A-asymptotic value

Table 7 Comparison of exact and asymptotic solutions for small values of H and fixed values of $\gamma = 0.01$, $\Lambda = 1.2$ and $\sigma^2 = 5$ in case of rigid–rigid boundaries

$\log_{10} H$	$v_c(E)$	$v_c(A)$	$Ra_{nlc}(E)$	$Ra_{nlc}(A)$
− 3.5	3.15664	3.15664	3214.38	3214.38
− 3	3.15668	3.15668	3214.49	3214.49
− 2.5	3.15682	3.15682	3214.84	3214.84
− 2	3.15724	3.15724	3215.95	3215.95
− 1.5	3.15856	3.15856	3219.45	3219.45
− 1	3.16273	3.16273	3230.52	3230.52
− 0.5	3.17573	3.17572	3265.41	3265.41
0	3.21515	3.21498	3374.81	3374.81
0.5	3.32557	3.32073	3712.55	3712.56

E-exact value, A-asymptotic value

The eigen solutions of the Eqs. (11)–(13) are:

$$\left. \begin{aligned} \Psi &= A \sin(vX) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right], \\ \Theta_{nl} &= B \cos(vX) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right], \\ \Theta_s &= C \cos(vX) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right], \end{aligned} \right\}, \quad (15)$$

where A , B and C are the amplitudes and v is the wave number. Substituting Eq. (15) in the linearised version of Eqs. (11)–(13) and following the orthogonalisation procedure and rearrangement, we get

$$Ra_{nlc}^{FF} = \frac{\delta_c^4 (\delta_c^2 \Lambda + \sigma^2)}{v_c^2} \left(1 + \frac{H}{\delta_c^2 + \gamma H} \right), \quad (16)$$

where $\delta_c^2 = v_c^2 + \pi^2$.

Table 8 Comparison of exact and asymptotic solutions for large values of H and fixed values of $\gamma = 5$, $\Lambda = 1.2$ and $\sigma^2 = 5$ in case of rigid–rigid boundaries

$\log_{10} H$	$v_c(E)$	$v_c(A)$	$Ra_{nlc}(E)$	$Ra_{nlc}(A)$
2	3.16415	3.16413	3832.64	3832.67
2.5	3.15911	3.15911	3849.22	3849.23
3	3.15742	3.15742	3854.65	3854.65
3.5	3.15688	3.15688	3856.39	3856.39
4	3.1567	3.1567	3856.94	3856.94
4.5	3.15665	3.15665	3857.11	3857.11
5	3.15663	3.15663	3857.17	3857.17
5.5	3.15662	3.15662	3857.18	3857.18
6	3.15662	3.15662	3857.19	3857.19
6.5	3.15662	3.15662	3857.19	3857.19
7	3.15662	3.15662	3857.19	3857.19
7.5	3.15662	3.15662	3857.19	3857.19
8	3.15662	3.15662	3857.19	3857.19

E-exact value, A-asymptotic value

Table 9 Comparison of exact and asymptotic solutions for large values of γ and fixed values of $H = 5$, $\Lambda = 1.2$ and $\sigma^2 = 5$ in case of rigid–rigid boundaries

$\log_{10} \gamma$	$v_c(E)$	$v_c(A)$	$Ra_{nlc}(E)$	$Ra_{nlc}(A)$
1	3.17998	3.20489	3482.47	3407.31
1.5	3.16034	3.16145	3309.97	3303.21
2	3.15706	3.1571	3245.84	3245.19
2.5	3.15667	3.15667	3224.43	3224.36
3	3.15663	3.15663	3217.53	3217.53
3.5	3.15662	3.15662	3215.34	3215.34
4	3.15662	3.15662	3214.65	3214.65
4.5	3.15662	3.15662	3214.43	3214.43
5	3.15662	3.15662	3214.36	3214.36
5.5	3.15662	3.15662	3214.34	3214.34
6	3.15662	3.15662	3214.33	3214.33

E-exact value, A-asymptotic value

Asymptotic Analysis

Case 1: For small values of H

We expand Ra given by Eq. (16) in a power series in H as

$$Ra_{nl}^{FF} = \frac{\delta^4 (\delta^2 \Lambda + \sigma^2)}{v^2} + \frac{\delta^2 (\delta^2 \Lambda + \sigma^2)}{v^2} H - \frac{\gamma (\delta^2 \Lambda + \sigma^2)}{v^2} H^2 + \dots, \quad (17)$$

where

$$\delta^2 = v^2 + \pi^2, \quad (18)$$

Setting $\frac{\partial Ra_{nl}^{FF}}{\partial v} = 0$ for minimizing Ra_{nl}^{FF} , we get:

$$4\Lambda v^6 + 2v^4 (3\pi^2 \Lambda + \sigma^2) - 2\pi^4 (\pi^2 \Lambda + \sigma^2) + (2\Lambda v^4 - 2\pi^2 (\pi^2 \Lambda + \sigma^2)) H + 2\gamma (\pi^2 \Lambda + \sigma^2) H^2 = 0 \quad (19)$$

Table 10 Comparison of exact and asymptotic solutions for large values of γH and fixed values of $H = 10$, $\Lambda = 1.2$ and $\sigma^2 = 5$ in case of rigid–rigid boundaries

$\log_{10} \gamma H$	$v_c(E)$	$v_c(A)$	$Ra_{nlc}(E)$	$Ra_{nlc}(A)$
2	3.17227	3.18075	3482.45	3471.78
2.5	3.1587	3.15904	3309.97	3309.59
3	3.15685	3.15686	3245.84	3245.83
3.5	3.15665	3.15665	3224.43	3224.43
4	3.15662	3.15662	3217.53	3217.53
4.5	3.15662	3.15662	3215.34	3215.34
5	3.15662	3.15662	3214.65	3214.65
5.5	3.15662	3.15662	3214.43	3214.43
6	3.15662	3.15662	3214.36	3214.36
6.5	3.15662	3.15662	3214.34	3214.34
7	3.15662	3.15662	3214.33	3214.33
7.5	3.15662	3.15662	3214.33	3214.33
8	3.15662	3.15662	3214.33	3214.33

E-exact value, A-asymptotic value

Fig. 6 Variation of weighted average Nusselt number, Nu_w with Rayleigh number, Ra_{nl} , for different values of H , for fixed values of $\Lambda = 1.2$, $\sigma^2 = 5$ and $\gamma = 0.1$

We also expand v in terms of H as

$$v = v_0 + v_1 H + v_2 H^2 + \dots, \quad (20)$$

where v_0 is the critical wave number for the LTE case which is given by

$$v_0 = \frac{1}{2} \sqrt{\frac{\sqrt{(\pi^2 \Lambda + \sigma^2)(9\pi^2 \Lambda + \sigma^2)} - \pi^2 \Lambda - \sigma^2}{\Lambda}}. \quad (21)$$

Substituting Eq. (20) in Eq. (19) and equating the coefficients of like powers of H , we obtain v_1 and v_2 as:

$$v_1 = \frac{\Delta'_1}{\Delta'_2}, \quad (22)$$

Fig. 7 Variation of Nu_w with Rayleigh number, Ra_{nl} , for different values of H , for fixed values of $\Lambda = 1.2$, $\sigma^2 = 5$ and $\gamma = 0.1$

Fig. 8 Variation of Nu_w with Ra_{nl} , for different values of γ , for fixed values of $\Lambda = 1.2$, $\sigma^2 = 5$ and $H = 10$

Fig. 9 Variation of Nu_w with Ra_{nl} , for different values of γ , for fixed values of $\Lambda = 1.2$, $\sigma^2 = 5$ and $H = 10$

$$v_2 = \frac{\Delta'_3}{\Delta'_2} \quad (23)$$

where

$$\delta_0^2 = v_0^2 + \pi^2, \quad (24)$$

$$\Delta'_1 = \Lambda (\pi^4 - v_0^4) + \pi^2 \sigma^2 \quad (25)$$

$$\Delta'_2 = 4v_0^3 (3\delta_0^2 \Lambda + \sigma^2) \quad (26)$$

$$\Delta'_3 = \Lambda (-\pi^2 \gamma - 2v_1 (9\pi^2 v_1 + v_0 (15v_0 v_1 + 2)) v_0^2) + \sigma^2 (\gamma + 6v_0^2 v_1^2) \quad (27)$$

Using the obtained values of v_0 , v_1 , v_2 in Eq. (20), we get the critical wave number and using this in the Eq. (17), we can obtain the critical Rayleigh number for small H .

Case 2: For large values of H

For large values of H , Ra_{nl}^{FF} takes the form

$$Ra_{nl}^{FF} = \frac{(\gamma + 1)\delta^4 (\delta^2 \Lambda + \sigma^2)}{\gamma v^2} + \frac{\delta^8 (\delta^2 \Lambda + \sigma^2)}{\gamma^3 H^2 v^2} - \frac{\delta^6 (\delta^2 \Lambda + \sigma^2)}{\gamma^2 H v^2} + \dots, \quad (28)$$

where $\delta^2 = v^2 + \pi^2$.

Setting $\frac{\partial Ra_{nl}^{FF}}{\partial v} = 0$, we get:

$$\begin{aligned} & 2\gamma^2(\gamma + 1) (2\Lambda v^6 + v^4 (3\pi^2 \Lambda + \sigma^2) - \pi^4 (\pi^2 \Lambda + \sigma^2)) \\ & + 2\gamma\delta^4 (\delta^2 \Lambda (\pi^2 - 3v^2) + (\pi^2 - 2v^2) \sigma^2) \frac{1}{H} \\ & + 2\delta^6 (4\Lambda v^4 + 3v^2 (\pi^2 \Lambda + \sigma^2) - \pi^2 (\pi^2 \Lambda + \sigma^2)) \frac{1}{H^2} = 0 \end{aligned} \quad (29)$$

We expand v in terms of $\frac{1}{H} \ll 1$ as

$$v = v_0 + \frac{v_1}{H} + \frac{v_2}{H^2} + \dots, \quad (30)$$

where v_0 is the critical wave number for the LTE case which is given by Eq. (21).

Substituting Eq. (30) in Eq. (29) and equating the coefficients of like powers of H , we obtain the values of v_1 and v_2 as:

$$v_1 = \frac{\delta_0^4 \Delta'_4}{\Delta'_5}, \quad (31)$$

$$v_2 = \frac{\Delta'_6}{\gamma \Delta'_5}, \quad (32)$$

where

$$\Delta'_4 = (2v_0^2 (\pi^2 \Lambda + \sigma^2) + 3\Lambda v_0^4 - \pi^2 (\pi^2 \Lambda + \sigma^2)), \quad (33)$$

$$\Delta'_5 = 4\gamma(\gamma + 1)v_0^3 (3\delta_0^2 \Lambda + \sigma^2). \quad (34)$$

$$\begin{aligned} \Delta'_6 = & -6\gamma^2(\gamma + 1)v_0^2 v_1^2 (5\Lambda v_0^2 + 3\pi^2 \Lambda + \sigma^2) \\ & + 12\gamma\delta_0^2 v_0^3 v_1 (2\delta_0^2 \Lambda + \sigma^2) + \delta_0^6 (\delta_0^2 \Lambda (\pi^2 - 4v_0^2) + (\pi^2 - 3v_0^2) \sigma^2) \end{aligned} \quad (35)$$

Table 11 Exact meaning of $H \rightarrow 0$ and $H \rightarrow \infty$, for large values of γ , and $\Lambda = 1.2$ and $\sigma^2 = 5$

γ	Free-free		Rigid-rigid	
	$H \rightarrow 0$ (LTE)	$H \rightarrow \infty$ (LTE)	$H \rightarrow 0$ (LTE)	$H \rightarrow \infty$ (LTE)
10	0.057	218.53	0.057	182.33
20	0.051	110.53	0.043	94.46
30	0.045	73.97	0.05	63.69
40	0.024	55.58	0.038	48.03
50	0.017	44.52	0.023	38.55
60	0.014	37.13	0.017	32.20
70	0.011	31.84	0.014	27.64
80	0.009	27.87	0.012	24.21
90	0.008	24.78	0.01	21.54
100	0.007	22.31	0.009	19.4

Table 12 Exact meaning of $\gamma \rightarrow \infty$ and $\gamma H \rightarrow \infty$, for large values of H , and $\Lambda = 1.2$ and $\sigma^2 = 5$

H	Free-free		Rigid-rigid	
	$\gamma \rightarrow \infty$ (LTE)	$\gamma H \rightarrow \infty$ (LTE)	$\gamma \rightarrow \infty$ (LTE)	$\gamma H \rightarrow \infty$ (LTE)
10	239.06	2390.56	229.3	2292.98
20	126.38	2527.56	127.3	2554.58
30	88.37	2651.10	92.38	2771.38
40	69.11	2764.07	73.97	2959.70
50	57.37	2868.48	62.50	3124.88
60	49.43	2965.78	54.59	3275.02
70	43.68	3057.08	48.76	3412.51
80	39.30	3143.21	44.25	3539.71
90	35.84	3224.85	40.65	3658.36
100	33.03	3302.55	37.7	3769.75

The critical wave number v_c is computed from the obtained values of v_0 , v_1 and v_2 and using this value of v_c we can obtain critical Rayleigh number from Eq. (28) for large H .

Case 3: For large values of γ

For large values of γ , i. e., $\frac{1}{\gamma} \ll 1$, Ra_{nl}^{FF} takes the form,

$$Ra_{nl}^{FF} = \frac{\delta^4 (\delta^2 \Lambda + \sigma^2)}{v^2} + \frac{\delta^4 (\delta^2 \Lambda + \sigma^2)}{v^2} \frac{1}{\gamma} - \frac{\delta^6 (\delta^2 \Lambda + \sigma^2)}{H v^2} \frac{1}{\gamma^2} + \dots, \quad (36)$$

where $\delta^2 = v^2 + \pi^2$.

We minimize Eq. (36) with respect to v and obtain the expression:

$$H (4\Lambda v^6 + 2v^4 (3\pi^2 \Lambda + \sigma^2) - 2\pi^4 (\pi^2 \Lambda + \sigma^2)) + H (4\Lambda v^6 + 2v^4 (3\pi^2 \Lambda + \sigma^2) - 2\pi^4 (\pi^2 \Lambda + \sigma^2)) \frac{1}{\gamma}$$

Fig. 10 Variation of Nu_w with Ra_{nl} , for different values of Λ , for fixed values of $\sigma^2 = 5$, $\gamma = 1$ and $H = 10$

$$+2\delta^4 (\delta^2 \Lambda (\pi^2 - 3v^2) + (\pi^2 - 2v^2) \sigma^2) \frac{1}{\gamma^2} = 0 \quad (37)$$

We expand v in the form,

$$v = v_0 + \frac{v_1}{\gamma} + \frac{v_2}{\gamma^2} + \dots, \quad (38)$$

where v_0 is the critical wave number for the LTE case given by the Eq. (21).

Substituting Eq. (38) in Eq. (36) and equating the coefficients of like powers of γ , we obtain the values of v_1 and v_2 as:

$$v_1 = \frac{\Delta'_7}{\Delta'_8}, \quad (39)$$

$$v_2 = \frac{\Delta'_9}{H \Delta'_8}, \quad (40)$$

where

$$\Delta'_7 = \delta_0^4 \Lambda (\pi^2 - 2v_0^2) + (\pi^4 - v_0^4) \sigma^2, \quad (41)$$

$$\Delta'_8 = 4v_0^3 (3\delta_0^2 \Lambda + \sigma^2), \quad (42)$$

$$\Delta'_9 = \delta_0^4 (2v_0^2 (\pi^2 \Lambda + \sigma^2) + 3\Lambda v_0^4 - \pi^2 (\pi^2 \Lambda + \sigma^2)) - 2H v_0^2 v_1 (2v_0 (3\Lambda v_0^2 + 3\pi^2 \Lambda + \sigma^2) + 3v_1 (5\Lambda v_0^2 + 3\pi^2 \Lambda + \sigma^2)). \quad (43)$$

The critical wave number v_c is computed from the obtained values of v_0 , v_1 and v_2 and using this value of v_c we can obtain critical Rayleigh number from Eq. (36) for large γ .

Case 4: For large values of γH

For large values of γH , i. e. $\frac{1}{\gamma H} \ll 1$, Ra_{nl}^{FF} takes the form,

$$Ra_{nl}^{FF} = \frac{\delta^4 (\delta^2 \Lambda + \sigma^2)}{v^2} + \frac{\delta^4 H (\delta^2 \Lambda + \sigma^2)}{v^2} \frac{1}{\gamma H} - \frac{\delta^6 H (\delta^2 \Lambda + \sigma^2)}{v^2} \frac{1}{(\gamma H)^2} + \dots, \quad (44)$$

where $\delta^2 = v^2 + \pi^2$.

Fig. 11 Variation of Nu_w with Ra_{nl} , for different values of Λ , for fixed values of $\sigma^2 = 5$, $\gamma = 10$ and $H=5$

Fig. 12 Variation of Nu_w with Ra_{nl} , for different values of σ^2 , for fixed values of $\Lambda = 1.2$, $\gamma = 1$ and $H = 10$

Fig. 13 Variation of Nu_w with Ra_{nl} , for different values of σ^2 , for fixed values of $\Lambda = 1.2$, $\gamma = 10$ and $H = 5$

Fig. 14 Variation of Nu_w with Ra_{nl} of LTNE and LTE for $\Lambda = 1.2$, $\sigma^2 = 10$ and $\gamma = 1$

Fig. 15 Variation of Nu_w with Ra_{nl} of LTNE and LTE for $\Lambda = 1.2$, $\sigma^2 = 10$ and $\gamma = 1$

Fig. 16 Variation of Nu_w with Ra_{nl} of free–free and rigid–rigid boundaries, for $\Lambda = 1.2$, $\sigma^2 = 5$, $\gamma = 5$ and $H = 10$

We minimize Eq. (44) with respect to v and obtain the expression:

$$\begin{aligned} & 4\Lambda v^6 + 2v^4 (3\pi^2 \Lambda + \sigma^2) - 2\pi^4 (\pi^2 \Lambda + \sigma^2) \\ & + 2H (2\Lambda v^6 + v^4 (3\pi^2 \Lambda + \sigma^2) - \pi^4 (\pi^2 \Lambda + \sigma^2)) \frac{1}{\gamma H} \\ & + 2\delta^4 H (\delta^2 \Lambda (\pi^2 - 3v^2) + (\pi^2 - 2v^2) \sigma^2) \frac{1}{(\gamma H)^2} = 0 \end{aligned} \quad (45)$$

We expand v in the form,

$$v = v_0 + \frac{v_1}{\gamma H} + \frac{v_2}{(\gamma H)^2} + \dots \quad (46)$$

where v_0 is given by Eq. (21).

Substituting Eq. (46) in Eq. (44) and equating the coefficients of like powers of γ , we obtain the values of v_1 and v_2 as:

$$v_1 = \frac{\Delta'_{10}}{\Delta'_{11}}, \quad (47)$$

$$v_2 = \frac{2\Delta'_{12}}{\Delta'_{11}}, \quad (48)$$

where

$$\Delta'_{10} = H\delta_0^4 \Lambda (\pi^2 - 2v_0^2) + H (\pi^4 - v_0^4) \sigma^2, \quad (49)$$

$$\Delta'_{11} = 4v_0^3 (3\delta_0^2 \Lambda + \sigma^2), \quad (50)$$

$$\begin{aligned} \Delta'_{12} = & \delta_0^2 H \Lambda (3v_0^6 + 5\pi^2 v_0^4 - 12v_1 v_0^3 + \pi^4 v_0^2 - \pi^6) \\ & + H (2v_0^6 + 3\pi^2 v_0^4 - 4v_1 v_0^3 - \pi^6) \sigma^2 - 6v_0^2 v_1^2 (5\Lambda v_0^2 + 3\pi^2 \Lambda + \sigma^2). \end{aligned} \quad (51)$$

The critical wave number v_c is computed from the obtained values of v_0 , v_1 and v_2 and using this value of v_c , we can obtain critical Rayleigh number from Eq. (44) for large γH .

Rigid–Rigid, Isothermal Boundaries

The boundary conditions suitable for rigid–rigid isothermal boundaries are:

$$\Psi = \frac{\partial \Psi}{\partial Z} = \Theta_{nl} = \Theta_s \quad \text{at } Z = \pm \frac{1}{2}. \quad (52)$$

The eigen solutions of the Eqs. (11)–(13) are:

$$\left. \begin{aligned} \Psi &= A \sin(vX) C_f(Z), \\ \Theta_{nl} &= B \cos(vX) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right], \\ \Theta_s &= C \cos(vX) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right], \end{aligned} \right\} \quad (53)$$

where

$$C_f(Z) = \frac{\cosh(\mu_1 Z)}{\cosh\left(\frac{\mu_1}{2}\right)} - \frac{\cos(\mu_1 Z)}{\cos\left(\frac{\mu_1}{2}\right)}$$

is the Chandrasekhar function [6,7,10] and $\mu_1 = 4.73004074$. Following the procedure used for free–free, isothermal boundaries, we can get the expression for the critical value of nanoliquid Rayleigh number for rigid–rigid, isothermal boundaries as:

$$Ra_{nl}^{RR} = \frac{\delta_c^2 Q_1}{2F_3^2 v_c^2} \left(1 + \frac{H}{\delta_c^2 + \gamma H} \right), \quad (54)$$

where

$$\delta_c^2 = v_c^2 + \pi^2, \quad (55)$$

$$Q_1 = F_1 \Lambda (v_c^4 + \mu_1^4) + F_1 v_c^2 \sigma^2 + F_2 \mu_1^2 (2v_c^2 \Lambda + \sigma^2), \quad (56)$$

$$F_1 = \frac{1}{1 + \cos \mu_1} + \frac{1}{1 + \cosh \mu_1} - \frac{\tan\left(\frac{\mu_1}{2}\right)}{\mu_1} - \frac{\tanh\left(\frac{\mu_1}{2}\right)}{\mu_1}, \quad (57)$$

$$F_2 = \frac{1}{1 + \cos \mu_1} - \frac{1}{1 + \cosh \mu_1} + \frac{\tan\left(\frac{\mu_1}{2}\right)}{\mu_1} - \frac{\tanh\left(\frac{\mu_1}{2}\right)}{\mu_1}, \quad (58)$$

$$F_3 = \frac{4\pi \mu_1^2}{\pi^4 - \mu_1^4}. \quad (59)$$

Asymptotic Analysis

Case 1: For small values of H

We expand Ra given by Eq. (54) in a power series in H as

$$Ra_{nl}^{RR} = \frac{\delta^2 Q_1}{2v^2 F_3^2} + \frac{Q_1}{2v^2 Q_3} H - \frac{\gamma Q_1}{2v^2 \delta^2 F_3^2} H^2 + \dots, \quad (60)$$

where Q_1 and F_3 are given by Eqs. (56) and (59) respectively and $\delta^2 = v^2 + \pi^2$.

Setting $\frac{\partial Ra_{nl}^{RR}}{\partial v} = 0$ for minimizing Ra_{nl}^{RR} , we get:

$$\begin{aligned} & \delta^4 (F_2 \mu_1^2 (2\Lambda v^4 - \pi^2 \sigma^2) + F_1 (v^4 (2\Lambda v^2 + \pi^2 \Lambda + \sigma^2) - \pi^2 \Lambda \mu_1^4)) \\ & + \delta^4 (F_1 \Lambda (v^4 - \mu_1^4) - F_2 \mu_1^2 \sigma^2) H + \gamma (F_2 \mu_1^2 (2\Lambda v^4 + (2v^2 + \pi^2) \sigma^2) \\ & + F_1 (\Lambda \mu_1^4 (2v^2 + \pi^2) + v^4 (\sigma^2 - \pi^2 \Lambda))) H^2 = 0. \end{aligned} \quad (61)$$

We expand v in terms of H as

$$v = v_0 + v_1 H + v_2 H^2 + \dots, \quad (62)$$

where v_0 is the critical wave number for the LTE case given by,

$$v_0 = \sqrt{\frac{Q_2^2 - Q_2 Q_3 + Q_3^2}{6\Lambda F_1 Q_3}} \quad (63)$$

where

$$Q_2 = F_1 (\pi^2 \Lambda + \sigma^2) + 2F_2 \Lambda \mu_1^2, \quad (64)$$

$$\begin{aligned} Q_3 = & \left(F_1^3 (54\pi^2 \Lambda^3 \mu_1^4 - (\pi^2 \Lambda + \sigma^2)^3) - 8F_2^3 \Lambda^3 \mu_1^6 - 12F_1 F_2^2 \Lambda^2 \mu_1^4 \right. \\ & \left. (\pi^2 \Lambda + \sigma^2) - 6F_1^2 F_2 \Lambda \mu_1^2 (\pi^4 \Lambda^2 - 7\pi^2 \Lambda \sigma^2 + \sigma^4) + 6\sqrt{3}\pi (-F_1^2 \Lambda^2 \mu_1^2 \right. \end{aligned}$$

$$\begin{aligned} & (F_1 \Lambda \mu_1^2 + F_2 \sigma^2) (8F_2^3 \Lambda^3 \mu_1^6 + 3F_1 \Lambda^2 \mu_1^4 (4F_2^2 (\pi^2 \Lambda + \sigma^2) - 9\pi^2 F_1^2 \Lambda) \\ & + 3F_1^2 F_2 \Lambda \mu_1^2 (2\pi^4 \Lambda^2 - 5\pi^2 \Lambda \sigma^2 + 2\sigma^4) + F_1^3 (\pi^2 \Lambda + \sigma^2)^3) \Big)^{1/2} \Big)^{1/3} \end{aligned} \quad (65)$$

and F_1 is given by Eq. (57).

Substituting Eq. (62) in Eq. (61) and equating the coefficients of like powers of H , we obtain v_1 and v_2 as:

$$v_1 = -\frac{\delta_0^2 \Delta_1}{\Delta_2}, \quad (66)$$

$$v_2 = \frac{\Delta_3}{\delta_0^2 \Delta_2}, \quad (67)$$

where

$$\delta_0^2 = v_0^2 + \pi^2, \quad (68)$$

$$\Delta_1 = F_1 \Lambda (v_0^4 - \mu_1^4) - F_2 \mu_1^2 \sigma^2, \quad (69)$$

$$\begin{aligned} \Delta_2 = & 4v_0 (F_2 \mu_1^2 (4\Lambda v_0^4 + 2\pi^2 \Lambda v_0^2 - \pi^2 \sigma^2) + F_1 (v_0^2 (\Lambda (5v_0^4 + 5\pi^2 v_0^2 + \pi^4) \\ & + (2v_0^2 + \pi^2) \sigma^2) - \pi^2 \Lambda \mu_1^4)), \end{aligned} \quad (70)$$

$$\begin{aligned} \Delta_3 = & F_2 \mu_1^2 (\sigma^2 (- (\gamma (2v_0^2 + \pi^2) - 2v_1 (2v_0^3 + \pi^2 (3v_0 v_1 + 2) v_0 + \pi^4 v_1))) \\ & - 2\Lambda v_0^2 (\gamma v_0^2 + 2 (14v_0^4 + 15\pi^2 v_0^2 + 3\pi^4) v_1^2)) - F_1 (\Lambda \mu_1^4 (\gamma (2v_0^2 + \pi^2) \\ & - 2v_1 (2v_0^3 + \pi^2 (3v_0 v_1 + 2) v_0 + \pi^4 v_1)) + v_0^2 (\Lambda (2v_1 ((45v_0 v_1 + 4) v_0^5 \\ & + 2\pi^2 (35v_0 v_1 + 3) v_0^3 + 2\pi^4 (15v_0 v_1 + 1) v_0 + 3\pi^6 v_1) - \pi^2 \gamma v_0^2) \\ & + \sigma^2 (\gamma v_0^2 + 2 (14v_0^4 + 15\pi^2 v_0^2 + 3\pi^4) v_1^2))). \end{aligned} \quad (71)$$

Using the obtained values of v_0 , v_1 , v_2 in Eq. (62) we get the critical wave number and using this in the Eq. (60) we can obtain the critical Rayleigh number for small H .

Case 2: For large values of H

For large values of H , Ra_{nl}^{RR} takes the form

$$Ra_{nl}^{RR} = \frac{\delta^2 (1 + \gamma) Q_1}{2v^2 \gamma F_3^2} - \frac{\delta^4 Q_1}{2v^2 \gamma^2 F_3^2} \frac{1}{H} + \frac{\delta^6 Q_1}{2v^2 \gamma^3 F_3^2} \frac{1}{H^2} + \dots, \quad (72)$$

where Q_1 and F_3 are given by Eq. (56) and Eq. (59) respectively and $\delta^2 = v^2 + \pi^2$.

Setting $\frac{\partial Ra_{nl}^{RR}}{\partial v} = 0$, we get:

$$\begin{aligned} & \gamma^2 (\gamma + 1) (F_2 \mu_1^2 (2\Lambda v^4 - \pi^2 \sigma^2) + F_1 (v^4 (2\Lambda v^2 + \pi^2 \Lambda + \sigma^2) - \pi^2 \Lambda \mu_1^4)) \\ & - (\gamma \delta^2 (F_2 \mu_1^2 (4\Lambda v^4 + v^2 \sigma^2 - \pi^2 \sigma^2) \\ & + F_1 (\Lambda \mu_1^4 (v - \pi)(v + \pi) + v^4 (3\Lambda v^2 + \pi^2 \Lambda + 2\sigma^2)))) \Big) \frac{1}{H} \\ & + (\delta^4 (F_2 \mu_1^2 (6\Lambda v^4 + (2v^2 - \pi^2) \sigma^2) + F_1 (\Lambda \mu_1^4 (2v^2 - \pi^2) \\ & + v^4 (4\Lambda v^2 + \pi^2 \Lambda + 3\sigma^2)))) \Big) \frac{1}{H^2} = 0 \end{aligned} \quad (73)$$

We expand ν in terms of $\frac{1}{H} \ll 1$ as

$$\nu = \nu_0 + \frac{\nu_1}{H} + \frac{\nu_2}{H^2} + \dots, \quad (74)$$

where ν_0 is the critical wave number for the LTE case given by Eq. (63).

Substituting Eq. (74) in Eq. (73) and equating the coefficients of like powers of H , we obtain the values of ν_1 and ν_2 as:

$$\nu_1 = \frac{\delta_0^2 \Delta_4}{\Delta_5}, \quad (75)$$

$$\nu_2 = \frac{\Delta_6}{\gamma \Delta_5} \quad (76)$$

where

$$\Delta_4 = F_1 (\Lambda \mu_1^4 (\nu_0^2 - \pi^2) + \nu_0^4 (3\Lambda \nu_0^2 + \pi^2 \Lambda + 2\sigma^2)) + F_2 \mu_1^2 (4\Lambda \nu_0^4 + \nu_0^2 \sigma^2 - \pi^2 \sigma^2), \quad (77)$$

$$\Delta_5 = 4\gamma(\gamma + 1)\nu_0^3 (2F_2 \Lambda \mu_1^2 + F_1 (3\Lambda \nu_0^2 + \pi^2 \Lambda + \sigma^2)), \quad (78)$$

$$\begin{aligned} \Delta_6 = & F_2 \mu_1^2 (\nu_0^2 (-12\gamma^2(\gamma + 1)\Lambda \nu_1^2 + 4\gamma \nu_1 \nu_0 (6\Lambda \nu_0^2 + 4\pi^2 \Lambda + \sigma^2) \\ & + \nu_0^2 (-6\delta_0^4 \Lambda + (2\nu_0^2 + 3\pi^2) \sigma^2))) + \pi^6 \sigma^2) + F_1 (\nu_0^2 (2\gamma \nu_1 (2\nu_0 \\ & (3\nu_0^2 (2\pi^2 \Lambda + \sigma^2) + 6\Lambda \nu_0^4 + \pi^4 \Lambda + 2\pi^2 \sigma^2) - 3\gamma(\gamma + 1)\nu_1 \\ & (5\Lambda \nu_0^2 + \pi^2 \Lambda + \sigma^2)) - \delta_0^4 \nu_0^2 (4\Lambda \nu_0^2 + \pi^2 \Lambda + 3\sigma^2)) + \Lambda \mu_1^4 (4\gamma \nu_1 \nu_0^3 \\ & - 2\nu_0^6 - 3\pi^2 \nu_0^4 + \pi^6)). \end{aligned} \quad (79)$$

The critical wave number ν_c is computed from the obtained values of ν_0 , ν_1 and ν_2 and using this value of ν_c we can obtain critical Rayleigh number from Eq. (72) for large H .

Case 3: For large values of γ

For large values of γ , i. e., $\frac{1}{\gamma} \ll 1$, Ra_{nl}^{RR} takes the form

$$Ra_{nl}^{RR} = \frac{\delta^2 Q_1}{2\nu^2 F_3^2} + \frac{\delta^2 Q_1}{2\nu^2 F_3^2} \frac{1}{\gamma} - \frac{\delta^4 Q_1}{2\nu^2 H F_3^2} \frac{1}{\gamma^2} + \dots, \quad (80)$$

where Q_1 and F_3 are given by Eq. (56) and Eq. (59) respectively and $\delta^2 = \nu^2 + \pi^2$.

We minimize Eq. (80) with respect to ν and obtain the expression:

$$\begin{aligned} & H (F_2 \mu_1^2 (2\Lambda \nu^4 - \pi^2 \sigma^2) + F_1 (\nu^4 (2\Lambda \nu^2 + \pi^2 \Lambda + \sigma^2) - \pi^2 \Lambda \mu_1^4)) \\ & + (H (F_2 \mu_1^2 (2\Lambda \nu^4 - \pi^2 \sigma^2) + F_1 (\nu^4 (2\Lambda \nu^2 + \pi^2 \Lambda + \sigma^2) - \pi^2 \Lambda \mu_1^4))) \frac{1}{\gamma} \\ & - (\delta^2 (F_2 \mu_1^2 (4\Lambda \nu^4 + \nu^2 \sigma^2 - \pi^2 \sigma^2) + F_1 (\Lambda \mu_1^4 (\nu^2 - \pi^2) \\ & + \nu^4 (3\Lambda \nu^2 + \pi^2 \Lambda + 2\sigma^2)))) \frac{1}{\gamma^2} = 0. \end{aligned} \quad (81)$$

We expand ν in the form,

$$\nu = \nu_0 + \frac{\nu_1}{\gamma} + \frac{\nu_2}{\gamma^2} + \dots, \quad (82)$$

where ν_0 is the critical wave number for the LTE case given by Eq. (63).

Substituting Eq. (82) in Eq. (80) and equating the coefficients of like powers of γ , we obtain the values of v_1 and v_2 as:

$$v_1 = \frac{\Delta_7}{\Delta_8}, \quad (83)$$

$$v_2 = \frac{\Delta_9}{H\Delta_8}, \quad (84)$$

where

$$\Delta_7 = F_1 (\pi^2 \Lambda \mu_1^4 - v_0^4 (2\Lambda v_0^2 + \pi^2 \Lambda + \sigma^2)) + F_2 \mu_1^2 (\pi^2 \sigma^2 - 2\Lambda v_0^4), \quad (85)$$

$$\Delta_8 = 4v_0^3 (2F_2 \Lambda \mu_1^2 + F_1 (3\Lambda v_0^2 + \pi^2 \Lambda + \sigma^2)), \quad (86)$$

$$\begin{aligned} \Delta_9 = & F_1 (v_0^2 (\delta_0^2 v_0^2 (3\Lambda v_0^2 + \pi^2 \Lambda + 2\sigma^2) - 4Hv_1 v_0 (3\Lambda v_0^2 + \pi^2 \Lambda + \sigma^2) \\ & - 6Hv_1^2 (5\Lambda v_0^2 + \pi^2 \Lambda + \sigma^2)) + \Lambda \mu_1^4 (v_0^4 - \pi^4)) + F_2 \mu_1^2 (4\Lambda v_0^2 \\ & (-2Hv_1 v_0 - 3Hv_1^2 + v_0^4 + \pi^2 v_0^2) + (v_0^4 - \pi^4) \sigma^2). \end{aligned} \quad (87)$$

The critical wave number v_c is computed from the obtained values of v_0 , v_1 and v_2 and using this value of v_c we can obtain critical Rayleigh number from Eq. (80) for large γ .

Case 4: For large values of γH

For large values of γH , i. e., $\frac{1}{\gamma H} \ll 1$, Ra_{nl}^{RR} takes the form

$$Ra_{nl}^{RR} = \frac{\delta^2 Q_1}{2v^2 F_3^2} + \frac{\delta^2 Q_1 H}{2v^2 F_3^2} \frac{1}{\gamma H} - \frac{\delta^4 H Q_1}{2v^2 F_3^2} \frac{1}{(\gamma H)^2} + \dots, \quad (88)$$

where Q_1 and F_3 are given by Eq. (56) and Eq. (59) respectively and $\delta^2 = v^2 + \pi^2$.

We minimize Eq. (88) with respect to v and obtain the expression:

$$\begin{aligned} & F_2 \mu_1^2 (2\Lambda v^4 - \pi^2 \sigma^2) + F_1 (v^4 (2\Lambda v^2 + \pi^2 \Lambda + \sigma^2) - \pi^2 \Lambda \mu_1^4) \\ & + (H (F_2 \mu_1^2 (2\Lambda v^4 - \pi^2 \sigma^2) + F_1 (v^4 (2\Lambda v^2 + \pi^2 \Lambda + \sigma^2) - \pi^2 \Lambda \mu_1^4))) \frac{1}{\gamma H} \\ & - (H (v^2 + \pi^2) (F_2 \mu_1^2 (4\Lambda v^4 + v^2 \sigma^2 - \pi^2 \sigma^2) \\ & + F_1 (\Lambda \mu_1^4 (v - \pi)(v + \pi) + v^4 (3\Lambda v^2 + \pi^2 \Lambda + 2\sigma^2)))) \frac{1}{(\gamma H)^2} = 0. \end{aligned} \quad (89)$$

We expand v in the form,

$$v = v_0 + \frac{v_1}{\gamma H} + \frac{v_2}{(\gamma H)^2} + \dots, \quad (90)$$

where v_0 is the critical wave number for the LTE case given by Eq. (63).

Substituting Eq. (90) in Eq. (88) and equating the coefficients of like powers of γ , we obtain the values of v_1 and v_2 as:

$$v_1 = \frac{\Delta_{10}}{\Delta_{11}}, \quad (91)$$

$$v_2 = \frac{\Delta_{12}}{\Delta_{11}}, \quad (92)$$

where

$$\Delta_{10} = H (F_2 \mu_1^2 (\pi^2 \sigma^2 - 2\Lambda v_0^4) + F_1 (\pi^2 \Lambda \mu_1^4 - v_0^4 (2\Lambda v_0^2 + \pi^2 \Lambda + \sigma^2))), \quad (93)$$

$$\Delta_{11} = 4v_0^3 (2F_2\Lambda\mu_1^2 + F_1 (3\Lambda v_0^2 + \pi^2\Lambda + \sigma^2)), \quad (94)$$

$$\begin{aligned} \Delta_{12} = & F_1 (v_0^2 (Hv_0 (\delta_0^2 v_0 (3\Lambda v_0^2 + \pi^2\Lambda + 2\sigma^2) - 4v_1 (3\Lambda v_0^2 + \pi^2\Lambda + \sigma^2)) \\ & - 6v_1^2 (5\Lambda v_0^2 + \pi^2\Lambda + \sigma^2)) + H\Lambda\mu_1^4 (v_0^4 - \pi^4)) + F_2\mu_1^2 (4\Lambda v_0^2 \\ & (Hv_0 (v_0^3 + \pi^2 v_0 - 2v_1) - 3v_1^2) + H (v_0^4 - \pi^4) \sigma^2). \end{aligned} \quad (95)$$

The critical wave number v_c is computed from the obtained values of v_0 , v_1 and v_2 and using this value of v_c we can obtain critical Rayleigh number from Eq. (88) for large γH .

The non-linear analysis will now be used to study heat transport.

Weakly Non-linear Stability Analysis for Steady Convection in the Case of Free–Free/Rigid–Rigid, Isothermal Boundaries

Free–Free, Isothermal Boundaries

The truncated representation for making a weakly non-linear analysis for free–free, isothermal boundaries is

$$\left. \begin{aligned} \Psi &= A \sin(v_c X) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right], \\ \Theta_{nl} &= B_1 \cos(v_c X) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right] + B_2 \sin \left[2\pi \left(Z + \frac{1}{2} \right) \right], \\ \Theta_s &= C_1 \cos(v_c X) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right] + C_2 \sin \left[2\pi \left(Z + \frac{1}{2} \right) \right]. \end{aligned} \right\}. \quad (96)$$

Substituting Eq. (96) into Eqs. (11)–(13) and using the orthogonality condition with the eigen functions on the resulting equations, we get the following algebraic equations:

$$a_1 B_1 k R a_{nl} + A \delta_c^2 (\Lambda \delta_c^2 + \sigma^2) = 0, \quad (97)$$

$$\frac{(1 + \pi B_2) v_c}{a_1 (H + \delta_c^2)} A + B_1 - \frac{H}{H + \delta_c^2} C_1 = 0, \quad (98)$$

$$A B_1 - \frac{2a_1 (H + 4\pi^2)}{\pi v_c} B_2 + \frac{2a_1 H}{\pi v_c} C_2 = 0, \quad (99)$$

$$B_1 - \frac{(\delta_c^2 + \gamma H)}{\gamma H} C_1 = 0, \quad (100)$$

$$B_2 - \frac{(H + 4\pi^2)}{\gamma H} C_2 = 0, \quad (101)$$

Solving Eqs. (97)–(101), we get

$$A^2 = \frac{8a_1^2 \delta_c^2 (H + 4\pi^2 + \gamma H) (H + \delta_c^2 + \gamma H) r}{v_c^2 (\gamma H + 4\pi^2) (\delta_c^2 + \gamma H)} \left[1 - \frac{1}{r} \right], \quad (102)$$

$$B_1 = -\frac{v_c (\delta_c^2 + \gamma H)}{a_1 \delta_c^2 (H + \delta_c^2 + \gamma H) r} A, \quad (103)$$

$$B_2 = -\frac{v_c^2 (\gamma H + 4\pi^2) (\delta_c^2 + \gamma H)}{8\pi a_1^2 \delta_c^2 (H + 4\pi^2 + \gamma H) (H + \delta_c^2 + \gamma H) r} A^2 = -\frac{1}{\pi} \left[1 - \frac{1}{r} \right], \quad (104)$$

$$C_1 = -\frac{\gamma H v_c}{a_1 r \delta_c^2 (\delta_c^2 + \gamma H + H)} A, \quad (105)$$

$$\begin{aligned} C_2 &= -\frac{\gamma H v_c^2 (\delta_c^2 + \gamma H)}{8\pi a_1^2 \delta_c^2 (H + 4\pi^2 + \gamma H) (H + \delta_c^2 + \gamma H) r} A^2 \\ &= -\frac{\gamma H}{\pi (\gamma H + 4\pi^2)} \left[1 - \frac{1}{r} \right], \end{aligned} \quad (106)$$

where

$$r = \frac{Ra_{nl}}{Ra_{nl_c}^{FF}} \quad (107)$$

is the scaled Rayleigh number.

Rigid–Rigid, Isothermal Boundaries

The truncated representation for making a steady, finite amplitude analysis for rigid–rigid, isothermal boundaries is

$$\left. \begin{aligned} \Psi &= A \sin(v_c X) C_f(Z), \\ \Theta_{nl} &= B_1 \cos(v_c X) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right] + B_2 \sin \left[2\pi \left(Z + \frac{1}{2} \right) \right], \\ \Theta_s &= C_1 \cos(v_c X) \sin \left[\pi \left(Z + \frac{1}{2} \right) \right] + C_2 \sin \left[2\pi \left(Z + \frac{1}{2} \right) \right]. \end{aligned} \right\} \quad (108)$$

Substituting Eq. (108) into Eqs. (11)–(13) and using orthogonalization condition with the eigen functions on the resulting equations, we get the following algebraic equations:

$$\begin{aligned} v_c \left(F_1 \left(\frac{\Lambda \mu_1^4}{v_c} + \sigma v_c + v_c^3 \Lambda \right) + F_2 \left(2\Lambda \mu_1^2 v_c + \frac{\mu_1^2 \sigma^2}{v_c} \right) \right) A \\ - a_1 F_3 v_c Ra_{nl} B_1 = 0, \end{aligned} \quad (109)$$

$$\frac{(2F_3 + F_4 B_2) v_c}{a_1 (H + \delta_c^2)} A - B_1 + \frac{H}{H + \delta_c^2} C_1 = 0, \quad (110)$$

$$AB_1 + \frac{2a_1 (H + 4\pi^2)}{v_c F_4} B_2 - \frac{2a_1 H}{v_c F_4} C_2 = 0, \quad (111)$$

$$B_1 - \frac{(\delta_c^2 + \gamma H)}{\gamma H} C_1 = 0, \quad (112)$$

$$B_2 - \frac{(H + 4\pi^2)}{\gamma H} C_2 = 0, \quad (113)$$

where F_1 , F_2 , F_3 are given by Eqs. (57)–(59) and

$$F_4 = \frac{16\pi^2 \mu_1^2 (39\pi^4 + \mu_1^4)}{81\pi^8 - 82\pi^4 \mu_1^4 + \mu_1^8}. \quad (114)$$

Solving Eqs. (109)–(113), we get

$$A^2 = \frac{8\pi^2 a_1^2 \delta_c^2 (H + 4\pi^2 + \gamma H) (H + \delta_c^2 + \gamma H) r}{F_4^2 v_c^2 (\gamma H + 4\pi^2) (\delta_c^2 + \gamma H)} \left[1 - \frac{1}{r} \right], \quad (115)$$

$$B_1 = \frac{2F_3 v_c (\delta_c^2 + \gamma H)}{a_1 \delta_c^2 (H + \delta_c^2 + \gamma H) r} A, \quad (116)$$

$$B_2 = -\frac{F_3 F_4 v_c^2 (\gamma H + 4\pi^2) (\delta_c^2 + \gamma H)}{4\pi^2 a_1^2 r \delta_c^2 (H + 4\pi^2 + \gamma H) (H + \delta_c^2 + \gamma H)} A^2 = -\frac{2F_3}{F_4} \left[1 - \frac{1}{r} \right], \quad (117)$$

$$C_1 = \frac{2\gamma F_3 H v_c}{a_1 \delta_c^2 (H + \delta_c^2 + \gamma H) r} A, \quad (118)$$

$$C_2 = -\frac{\gamma F_3 F_4 H v_c^2 (\delta_c^2 + \gamma H)}{4\pi^2 a_1^2 r \delta_c^2 (H + 4\pi^2 + \gamma H) (H + \delta_c^2 + \gamma H)} A^2 \\ = -\frac{2F_3 \gamma H}{F_4 (\gamma H + 4\pi^2)} \left[1 - \frac{1}{r} \right], \quad (119)$$

where

$$r = \frac{Ra_{nl}}{Ra_{nlc}^{RR}} \quad (120)$$

is the scaled Rayleigh number.

We next study heat transport in terms of the Nusselt number.

Nusselt Number

Free–Free, Isothermal Boundaries

The amount of heat transport by Rayleigh–Bénard convection for free–free, isothermal boundaries can be quantified in terms of the Nusselt number, Nu , as follows:

$$Nu = \frac{\text{Heat transport by (conduction + convection)}}{\text{Heat transport by conduction}}.$$

Using Fourier first law and further simplyfying, we may write the expression for the Nusselt number for the nanoliquid and solid phases as

$$Nu_{nl} = 1 + \frac{\left[-k_{nl} \int_0^{\frac{2\pi}{v}} \frac{\partial \Theta_{nl}}{\partial Z} dX \right]_{Z=-\frac{1}{2}}}{\left[-k_{nl} \int_0^{\frac{2\pi}{v}} \frac{d\Theta_{nlb}}{dZ} dX \right]_{Z=-\frac{1}{2}}}, \quad (121)$$

$$Nu_s = 1 + \frac{\left[-k_s \int_0^{\frac{2\pi}{v}} \frac{\partial \Theta_s}{\partial Z} dX \right]_{Z=-\frac{1}{2}}}{\left[-k_s \int_0^{\frac{2\pi}{v}} \frac{d\Theta_{sb}}{bZ} dX \right]_{Z=-\frac{1}{2}}}, \quad (122)$$

where Nu_{nl} is Nusselt number of the nanoliquid phase and Nu_s is that of the solid phase, $\Theta_{nlb} = \frac{T_{nlb}}{\Delta T}$ and $\Theta_{sb} = \frac{T_{sb}}{\Delta T}$.

The weighted-average Nusselt number, Nu_w , is given by

$$Nu_w = \phi Nu_{nl} + (1 - \phi) Nu_s. \quad (123)$$

Substituting Eqs. (7)–(8) and Eq. (96) in Eqs. (121) and (122) and completing the integration, we get

$$Nu_{nl} = 1 - 2\pi B_2, \quad (124)$$

$$Nu_s = 1 - 2\pi C_2. \quad (125)$$

Substituting Eq. (104) and Eq. (106) in Eqs. (124) and (125), we get

$$Nu_{nl}^{FF} = 1 + 2 \left[1 - \frac{1}{r} \right], \quad (126)$$

$$Nu_s^{FF} = 1 + 2 \left(\frac{\gamma H}{4\pi^2 + \gamma H} \right) \left[1 - \frac{1}{r} \right]. \quad (127)$$

Substituting Eqs. (126) and (127) in Eq. (123), we get

$$Nu_w^{FF} = 1 + 2 \left(\frac{4\pi^2 \phi + \gamma H}{4\pi^2 + \gamma H} \right) \left[1 - \frac{1}{r} \right]. \quad (128)$$

where r is given by Eq. (107).

Asymptotic Analysis for Small Values of H

For small values of H with Ra_{nl}^{FF} given by Eq. (17) and ν given by Eq. (20), Nu_w^{FF} takes the form:

$$Nu_w^{FF} = G_1 + G_2 H + G_3 H^2 + \dots, \quad (129)$$

where

$$G_1 = 1 + 2\phi \left(1 - \frac{\delta_0^2 (\delta_0^2 \Lambda + \sigma^2)}{v_0^2 Ra_{nl}} \right), \quad (130)$$

$$G_2 = \frac{1}{2\pi^2 v_0^3 Ra_{nl}} \left(v_0^5 (\pi^2 \Lambda (3\gamma (\phi - 1) - 4\phi) + \gamma \sigma^2 (\phi - 1)) + \pi^4 v_0 (\gamma (\phi - 1) - 4\phi) (\pi^2 \Lambda + \sigma^2) + \gamma \Lambda v_0^7 (\phi - 1) - 8\pi^2 v_1 v_0^4 \phi (3\pi^2 \Lambda + \sigma^2) + 8\pi^6 v_1 \phi (\pi^2 \Lambda + \sigma^2) - 16\pi^2 \Lambda v_1 v_0^6 \phi - v_0^3 (\pi^2 (\pi^2 \Lambda (-3\gamma \phi + 3\gamma + 8\phi) + 2\sigma^2 (-\gamma \phi + \gamma + 2\phi)) + \gamma Ra_{nl} (\phi - 1)) \right), \quad (131)$$

$$G_3 = \frac{1}{8\pi^4 v_0^4 Ra_{nl}} \left(v_0^6 (-\gamma^2 \sigma^2 (\phi - 1) - \pi^2 \gamma (3\gamma - 4) \Lambda (\phi - 1) - 96\pi^4 \Lambda v_1^2 \phi) - \gamma^2 \Lambda v_0^8 (\phi - 1) - 8\pi^2 v_0^5 (v_1 (\pi^2 \Lambda (-3\gamma \phi + 3\gamma + 4\phi) - \gamma \sigma^2 (\phi - 1)) + 4\pi^2 v_2 \phi (3\pi^2 \Lambda + \sigma^2)) - \pi^4 \gamma v_0^2 (\gamma (\phi - 1) - 20\phi + 4) (\pi^2 \Lambda + \sigma^2) + 8\pi^6 v_0 (\pi^2 \Lambda + \sigma^2) (v_1 (\gamma (-\phi) + \gamma + 4\phi) + 4\pi^2 v_2 \phi) + 16\pi^2 \Lambda v_0^7 (\gamma v_1 (\phi - 1) - 4\pi^2 v_2 \phi) - 48\pi^8 v_1^2 \phi (\pi^2 \Lambda + \sigma^2) + v_0^4 (\gamma^2 Ra_{nl} (\phi - 1) - \pi^2 (\pi^2 (3\gamma^2 \Lambda (\phi - 1) + 8\gamma \Lambda (1 - 3\phi) + 16v_1^2 \sigma^2 \phi) + 2(\gamma - 2)\gamma \sigma^2 (\phi - 1) + 48\pi^4 \Lambda v_1^2 \phi)) \right). \quad (132)$$

Asymptotic Analysis for Large Values of H , i. e., $\frac{1}{H} \ll 1$

For large values of H with Ra_{nl}^{FF} given by Eq. (28) and ν given by Eq. (30), Nu_w^{FF} takes the form:

$$Nu_w^{FF} = G_4 + G_5 \frac{1}{H} + G_6 \frac{1}{H^2} + \dots, \quad (133)$$

where

$$G_4 = 3 - \frac{2(\gamma + 1)\delta_0^4 (\delta_0^2 \Lambda + \sigma^2)}{\gamma v_0^2 Ra_{nl}}, \quad (134)$$

$$G_5 = \frac{1}{\gamma^2 v_0^3 Ra_{nl}} \left(2 \left(v_0^7 (\sigma^2 - 4\pi^2 \Lambda (\gamma(\phi - 1) + \phi - 2)) + \pi^2 v_0^5 (\sigma^2 \right. \right. \\ \left. \left. (-4(\gamma + 1)\phi + 4\gamma + 7) - 6\pi^2 \Lambda (2(\gamma + 1)\phi - 2\gamma - 3)) - \pi^6 v_0 (4\gamma(\phi - 1) \right. \right. \\ \left. \left. + 4\phi - 5) (\pi^2 \Lambda + \sigma^2) - 2\gamma(\gamma + 1)v_1 v_0^4 (3\pi^2 \Lambda + \sigma^2) + 2\pi^4 \gamma(\gamma + 1)v_1 \right. \right. \\ \left. \left. (\pi^2 \Lambda + \sigma^2) - 4\gamma(\gamma + 1)\Lambda v_1 v_0^6 + \Lambda v_0^9 + \pi^2 v_0^3 (\pi^2 (4(\gamma - (\gamma + 1)\phi) \right. \right. \right. \\ \left. \left. (3\pi^2 \Lambda + 2\sigma^2) + 11\sigma^2) + 16\pi^4 \Lambda + 4\gamma Ra_{nl}(\phi - 1)) \right) \right), \quad (135)$$

$$G_6 = -\frac{1}{\gamma^3 v_0^4 Ra_{nl}} \left(2 \left(2v_0^6 (3\gamma^2(\gamma + 1)\Lambda v_1^2 + \pi^6 \Lambda (-12(2\gamma + 3)\phi \right. \right. \\ \left. \left. + 24\gamma + 41) + \pi^4 \sigma^2 (-2(4\gamma + 7)\phi + 8\gamma + 17)) + 3\pi^4 \gamma^2(\gamma + 1)v_1^2 \right. \right. \\ \left. \left. (\pi^2 \Lambda + \sigma^2) + 2\pi^2 v_0^8 (\pi^2 \Lambda (-8(\gamma + 2)\phi + 8\gamma + 21) - 2\sigma^2(\phi - 2)) \right. \right. \\ \left. \left. + 4\gamma v_0^7 (v_1 (4\pi^2 \Lambda (\gamma(\phi - 1) + \phi - 2) - \sigma^2) + \gamma(\gamma + 1)\Lambda v_2) + 2\gamma v_0^5 \right. \right. \\ \left. \left. (\pi^2 v_1 (6\pi^2 \Lambda (2(\gamma + 1)\phi - 2\gamma - 3) + \sigma^2 (4(\gamma + 1)\phi - 4\gamma - 7)) \right. \right. \\ \left. \left. + \gamma(\gamma + 1)v_2 (3\pi^2 \Lambda + \sigma^2)) - \pi^8 v_0^2 (16\gamma(\phi - 1) + 20\phi - 21) (\pi^2 \Lambda + \sigma^2) \right. \right. \\ \left. \left. - 2\pi^4 \gamma v_0 (\pi^2 \Lambda + \sigma^2) (\pi^2 v_1 (4(\gamma + 1)\phi - 4\gamma - 5) + \gamma(\gamma + 1)v_2) \right. \right. \\ \left. \left. - 6\gamma \Lambda v_1 v_0^9 + v_0^{10} (\pi^2 \Lambda (9 - 4\phi) + \sigma^2) + \Lambda v_0^{12} + v_0^4 (3\pi^2 \gamma^2(\gamma + 1)\Lambda v_1^2 \right. \right. \\ \left. \left. + \gamma^2(\gamma + 1)v_1^2 \sigma^2 + \pi^8 \Lambda (-48\gamma(\phi - 1) - 64\phi + 69) + 4\pi^6 \sigma^2 (-8\gamma\phi + 8\gamma \right. \right. \\ \left. \left. - 11\phi + 12) + 16\pi^4 \gamma Ra_{nl}(\phi - 1)) \right) \right) \quad (136)$$

Asymptotic Analysis for Large Values of γ , i. e., $\frac{1}{\gamma} \ll 1$

For large values of γ with Ra_{nl}^{FF} given by Eq. (36) and ν given by Eq. (38), Nu_w^{FF} takes the form:

$$Nu_w^{FF} = G_7 + G_8 \frac{1}{\gamma} + G_9 \frac{1}{\gamma^2} + \dots, \quad (137)$$

where

$$G_7 = 3 - \frac{2\delta_0^4 (\delta_0^2 \Lambda + \sigma^2)}{v_0^2 Ra_{nl}}, \quad (138)$$

$$G_8 = -\frac{1}{H v_0^3 Ra_{nl}} \left(2 \left(v_0^5 (H + 4\pi^2(\phi - 1)) (3\pi^2 \Lambda + \sigma^2) + \pi^4 v_0 (H \right. \right. \\ \left. \left. + 4\pi^2(\phi - 1)) (\pi^2 \Lambda + \sigma^2) + 2H v_1 v_0^4 (3\pi^2 \Lambda + \sigma^2) - 2\pi^4 H v_1 \right. \right.$$

$$(\pi^2 \Lambda + \sigma^2) + \Lambda v_0^7 (H + 4\pi^2(\phi - 1)) + 4H \Lambda v_1 v_0^6 + \pi^2 v_0^3 ((H + 4\pi^2(\phi - 1)) (3\pi^2 \Lambda + 2\sigma^2) - 4Ra_{nl}(\phi - 1))), \quad (139)$$

$$G_9 = \frac{1}{H^2 v_0^4 Ra_{nl}} (2 (v_0^6 (-6H^2 \Lambda v_1^2 + \pi^2 H (6\pi^2 \Lambda (3 - 2\phi) + \sigma^2 (7 - 4\phi)) + 16\pi^4 (\phi - 1) (3\pi^2 \Lambda + \sigma^2)) - 3\pi^4 H^2 v_1^2 (\pi^2 \Lambda + \sigma^2) + v_0^4 (-3\pi^2 H^2 \Lambda v_1^2 - H^2 v_1^2 \sigma^2 - 4\pi^6 (H \Lambda (3\phi - 4) - 8\sigma^2 (\phi - 1)) + \pi^4 (H \sigma^2 (11 - 8\phi) - 16Ra_{nl}(\phi - 1)) + 48\pi^8 \Lambda (\phi - 1)) + v_0^8 (H (\sigma^2 - 4\pi^2 \Lambda (\phi - 2)) + 16\pi^4 \Lambda (\phi - 1)) - 2H v_0^5 (3\pi^2 \Lambda + \sigma^2) (H (v_1 + v_2) + 4\pi^2 v_1 (\phi - 1)) + \pi^6 v_0^2 (H (5 - 4\phi) + 16\pi^2 (\phi - 1)) (\pi^2 \Lambda + \sigma^2) + 2\pi^4 H v_0 (\pi^2 \Lambda + \sigma^2) (H (v_1 + v_2) + 4\pi^2 v_1 (\phi - 1)) - 4H \Lambda v_0^7 (H (v_1 + v_2) + 4\pi^2 v_1 (\phi - 1)) + H \Lambda v_0^{10})). \quad (140)$$

Asymptotic Analysis for Large Values of γH , i. e., $\frac{1}{\gamma H} \ll 1$

For large values of γH with Ra_{nl}^{FF} given by Eq. (44) and v given by Eq. (46), Nu_w^{FF} takes the form:

$$Nu_w^{FF} = G_{10} + G_{11} \frac{1}{\gamma H} + G_{12} \frac{1}{(\gamma H)^2} + \dots, \quad (141)$$

where $\delta_0^2 = v_0^2 + \pi^2$,

$$G_{10} = 3 - \frac{2\delta_0^4 (\delta_0^2 \Lambda + \sigma^2)}{v_0^2 Ra_{nl}}, \quad (142)$$

$$G_{11} = -\frac{1}{v_0^3 Ra_{nl}} (2 (v_0^5 (H + 4\pi^2(\phi - 1)) (3\pi^2 \Lambda + \sigma^2) + \pi^4 v_0 (H + 4\pi^2(\phi - 1)) (\pi^2 \Lambda + \sigma^2) + \Lambda v_0^7 (H + 4\pi^2(\phi - 1)) + \pi^2 v_0^3 ((H + 4\pi^2(\phi - 1)) (3\pi^2 \Lambda + 2\sigma^2) - 4Ra_{nl}(\phi - 1)) + 2v_1 v_0^4 (3\pi^2 \Lambda + \sigma^2) - 2\pi^4 v_1 (\pi^2 \Lambda + \sigma^2) + 4\Lambda v_1 v_0^6)), \quad (143)$$

$$G_{12} = \frac{1}{v_0^4 Ra_{nl}} (2 (v_0^8 (H (\sigma^2 - 4\pi^2 \Lambda (\phi - 2)) + 16\pi^4 \Lambda (\phi - 1)) + v_0^6 (\pi^2 (H (6\pi^2 \Lambda (3 - 2\phi) + \sigma^2 (7 - 4\phi)) + 16\pi^2 (\phi - 1) (3\pi^2 \Lambda + \sigma^2)) - 6\Lambda v_1^2) - 2v_0^5 (3\pi^2 \Lambda + \sigma^2) (H v_1 + 4\pi^2 v_1 (\phi - 1) + v_2) + \pi^6 v_0^2 (H (5 - 4\phi) + 16\pi^2 (\phi - 1)) (\pi^2 \Lambda + \sigma^2) + 2\pi^4 v_0 (\pi^2 \Lambda + \sigma^2) (H v_1 + 4\pi^2 v_1 (\phi - 1) + v_2) - 4\Lambda v_0^7 (H v_1 + 4\pi^2 v_1 (\phi - 1) + v_2) + H \Lambda v_0^{10} + v_0^4 (-4\pi^6 (H \Lambda (3\phi - 4) - 8\sigma^2 (\phi - 1)) + \pi^4 (H \sigma^2 (11 - 8\phi) - 16Ra_{nl}(\phi - 1)) - 3\pi^2 \Lambda v_1^2 + 48\pi^8 \Lambda (\phi - 1) + v_1^2 (-\sigma^2)) - 3\pi^4 v_1^2 (\pi^2 \Lambda + \sigma^2))). \quad (144)$$

Rigid–Rigid, Isothermal Boundaries

Following the procedure used for free–free, isothermal boundaries, we obtain the nanoliquid, solid and weighted average Nusselt number for rigid–rigid, isothermal boundaries as:

$$Nu_{nl}^{RR} = 1 + 4\pi \frac{F_3}{F_4} \left[1 - \frac{1}{r} \right], \quad (145)$$

$$Nu_s^{RR} = 1 + 4\pi \frac{F_3}{F_4} \left(\frac{\gamma H}{4\pi^2 + \gamma H} \right) \left[1 - \frac{1}{r} \right], \quad (146)$$

$$Nu_w^{RR} = 1 + 4\pi \frac{F_3}{F_4} \left(\frac{4\pi^2 \phi + \gamma H}{4\pi^2 + \gamma H} \right) \left[1 - \frac{1}{r} \right], \quad (147)$$

where F_3 , F_4 and r are given by Eqs. (59), (114) and (120).

Asymptotic Analysis for Small Values of H

For small values of H with Ra_{nl}^{RR} given by Eq. (60) and ν given by Eq. (62), Nu_w^{RR} takes the form:

$$Nu_w^{RR} = G_{13} + G_{14}H + G_{15}H^2 + \dots, \quad (148)$$

where

$$G_{13} = 1 + \frac{2\pi\phi(2F_3^2\nu_0^2Ra_{nl} - \delta_0^2Q_1)}{F_3F_4\nu_0^2Ra_{nl}}, \quad (149)$$

$$G_{14} = \frac{1}{2\pi F_3F_4\nu_0^3Ra_{nl}} \left(\phi(F_1\Lambda(\nu_0^4(\gamma\delta_0^2\nu_0 - 4\pi^2(4\nu_1\nu_0^2 + \nu_0 + 2\pi^2\nu_1)) + \mu_1^4(\gamma\nu_0^3 + \pi^2(\gamma - 4)\nu_0 + 8\pi^4\nu_1)) + F_2\mu_1^2(2\Lambda\nu_0^3(\gamma\nu_0^2 + \pi^2(\gamma - 8\nu_0\nu_1 - 4)) + \sigma^2(\gamma\nu_0^3 + \pi^2(\gamma - 4)\nu_0 + 8\pi^4\nu_1)) + F_1\nu_0^3\sigma^2(\gamma\nu_0^2 + \pi^2(\gamma - 8\nu_0\nu_1 - 4)) - \gamma\delta_0^2\nu_0(F_2\mu_1^2(2\Lambda\nu_0^2 + \sigma^2) + F_1\Lambda(\mu_1^4 + \nu_0^4) + F_1\nu_0^2\sigma^2) - 2\gamma F_3^2\nu_0^3Ra_{nl}(\phi - 1)) \right), \quad (150)$$

$$G_{15} = \frac{1}{8\pi F_3F_4\nu_0^4Ra_{nl}} \left(8\nu_1\nu_0(\nu_0^4(2\gamma F_2\Lambda\mu_1^2(\phi - 1) + F_1(\pi^2\Lambda((\gamma - 4)\phi - \gamma) + \gamma\sigma^2(\phi - 1))) - \pi^2\mu_1^2(\gamma(\phi - 1) - 4\phi)(F_1\Lambda\mu_1^2 + F_2\sigma^2) + 2\gamma F_1\Lambda\nu_0^6(\phi - 1)) - 16\pi^2\phi(F_2\mu_1^2(\nu_1^2(2\Lambda\nu_0^4 + 3\pi^2\sigma^2) + 2\nu_0\nu_2(2\Lambda\nu_0^4 - \pi^2\sigma^2)) + F_1(\pi^2\Lambda\mu_1^4(3\nu_1^2 - 2\nu_0\nu_2) + \nu_0^4(\nu_1^2(6\Lambda\nu_0^2 + \pi^2\Lambda + \sigma^2) + 2\nu_0\nu_2(2\Lambda\nu_0^2 + \pi^2\Lambda + \sigma^2)))) + \frac{1}{\pi^2}(\gamma\nu_0^2(2\gamma F_3^2\nu_0^2 Ra_{nl}(\phi - 1) - \frac{1}{\delta_0^2}((\nu_0^2(\phi - 1)(\gamma\nu_0^2 + 2\pi^2(\gamma - 2)) + \pi^4(\gamma(\phi - 1) - 20\phi + 4))(F_2\mu_1^2(2\Lambda\nu_0^2 + \sigma^2) + F_1(\Lambda(\mu_1^4 + \nu_0^4) + \nu_0^2\sigma^2)))))) \right). \quad (151)$$

Asymptotic Analysis for Large Values of H , i. e., $\frac{1}{H} \ll 1$

For large values of H with Ra_{nl}^{RR} given by Eq. (72) and ν given by Eq. (74), Nu_w^{RR} takes the form:

$$Nu_w^{RR} = G_{16} + G_{17} \frac{1}{H} + G_{18} \frac{1}{H^2} + \dots \quad (152)$$

where

$$G_{16} = 1 + \frac{2\pi (2\gamma F_3^2 v_0^2 Ra_{nl} - (\gamma + 1)\delta_0^2 Q_1)}{\gamma F_3 F_4 v_0^2 Ra_{nl}}, \quad (153)$$

$$G_{17} = \frac{1}{\gamma^2 F_3 F_4 Ra_{nl}} \left(2\pi \left(\frac{1}{v_0^2} (\delta_0^2 (\pi^2 (-4(\gamma + 1)\phi + 4\gamma + 5) + v_0^2) Q_1) - \frac{1}{v_0^3} (2\gamma(\gamma + 1)v_1 (F_2 \mu_1^2 (2\Lambda v_0^4 - \pi^2 \sigma^2) + F_1 (v_0^4 (2\Lambda v_0^2 + \pi^2 \Lambda + \sigma^2) - \pi^2 \Lambda \mu_1^4))) + 8\pi^2 \gamma F_3^2 Ra_{nl} (\phi - 1) \right) \right), \quad (154)$$

$$G_{18} = -\frac{1}{\gamma^3 F_3 F_4 v_0^4 Ra_{nl}} (2\pi (F_2 \mu_1^2 (v_0^4 (2\gamma^2 (\gamma + 1)\Lambda v_1^2 + 2\pi^6 \Lambda (-4(4\gamma + 5)\phi + 16\gamma + 21) + \pi^4 \sigma^2 (-8(2\gamma + 3)\phi + 16\gamma + 27)) + 3\pi^2 \gamma^2 (\gamma + 1)v_1^2 \sigma^2 + \pi^2 v_0^6 (2\pi^2 \Lambda (-8(2\gamma + 3)\phi + 16\gamma + 27) + \sigma^2 (7 - 4\phi)) + 2\gamma v_0^5 (v_1 (4\pi^2 \Lambda (2(\gamma + 1)\phi - 2\gamma - 3) - \sigma^2) + 2\gamma(\gamma + 1)\Lambda v_2) - 8\gamma \Lambda v_1 v_0^7 + \pi^6 v_0^2 \sigma^2 (-4(4\gamma + 5)\phi + 16\gamma + 21) - 2\pi^2 \gamma v_0 \sigma^2 (\pi^2 v_1 (4\gamma(\phi - 1) + 4\phi - 5) + \gamma(\gamma + 1)v_2) + v_0^8 (2\pi^2 \Lambda (7 - 4\phi) + \sigma^2) + 2\Lambda v_0^{10}) + F_1 (\Lambda \mu_1^4 (3\pi^2 \gamma^2 (\gamma + 1)v_1^2 + \pi^4 v_0^4 (-8(2\gamma + 3)\phi + 16\gamma + 27) + \pi^6 v_0^2 (-4(4\gamma + 5)\phi + 16\gamma + 21) + 2\pi^2 \gamma v_0 (\pi^2 v_1 (-4(\gamma + 1)\phi + 4\gamma + 5) - \gamma(\gamma + 1)v_2) - 2\gamma v_1 v_0^5 + \pi^2 v_0^6 (7 - 4\phi) + v_0^8) + v_0^4 (v_0^2 (6\gamma^2 (\gamma + 1)\Lambda v_1^2 + \pi^4 (\pi^2 \Lambda (-4(4\gamma + 5)\phi + 16\gamma + 21) + \sigma^2 (-8(2\gamma + 3)\phi + 16\gamma + 27))) + \gamma^2 (\gamma + 1)v_1^2 (\pi^2 \Lambda + \sigma^2) + \pi^2 v_0^4 (\pi^2 \Lambda (-8(2\gamma + 3)\phi + 16\gamma + 27) + \sigma^2 (7 - 4\phi)) + 4\gamma v_0^3 (v_1 (2\pi^2 \Lambda (2(\gamma + 1)\phi - 2\gamma - 3) - \sigma^2) + \gamma(\gamma + 1)\Lambda v_2) + 2\gamma v_0 (\pi^2 v_1 (\pi^2 \Lambda (4(\gamma + 1)\phi - 4\gamma - 5) + 2\sigma^2 (2(\gamma + 1)\phi - 2\gamma - 3)) + \gamma(\gamma + 1)v_2 (\pi^2 \Lambda + \sigma^2)) - 6\gamma \Lambda v_1 v_0^5 + \pi^6 \sigma^2 (-4(4\gamma + 5)\phi + 16\gamma + 21) + v_0^6 (\pi^2 \Lambda (7 - 4\phi) + \sigma^2) + \Lambda v_0^8)) + 32\pi^4 \gamma F_3^2 v_0^4 Ra_{nl} (\phi - 1))). \quad (155)$$

Asymptotic Analysis for Large Values of γ , i. e., $\frac{1}{\gamma} \ll 1$

For large values of γ with Ra_{nl}^{RR} given by Eq. (80) and ν given by Eq. (82), Nu_w^{RR} takes the form:

$$Nu_w^{RR} = G_{19} + G_{20} \frac{1}{\gamma} + G_{21} \frac{1}{\gamma^2} + \dots, \quad (156)$$

where

$$G_{19} = 1 + \frac{2\pi\phi(2F_3^2v_0^2Ra_{nl} - \delta_0^2Q_1)}{F_3F_4v_0^2Ra_{nl}}, \quad (157)$$

$$G_{20} = -\frac{1}{F_3F_4Hv_0^3Ra_{nl}} \left(2\pi(F_2\mu_1^2(v_0(2\Lambda v_0^2 + \sigma^2)(\pi^2(4\delta_0^2(\phi - 1) + H) + Hv_0^2) - 2Hv_1(\pi^2\sigma^2 - 2\Lambda v_0^4)) + F_1(v_0(\pi^2(4\delta_0^2(\phi - 1) + H) + Hv_0^2)(\Lambda(\mu_1^4 + v_0^4) + v_0^2\sigma^2) + 2Hv_1(-\pi^2\Lambda\mu_1^4 + v_0^4(\pi^2\Lambda + \sigma^2) + 2\Lambda v_0^6)) - 8\pi^2F_3^2v_0^3Ra_{nl}(\phi - 1)) \right), \quad (158)$$

$$G_{21} = -\frac{1}{F_3F_4H^2v_0^4Ra_{nl}} \left(2\pi(F_2\mu_1^2(H(v_1(Hv_1(2\Lambda v_0^4 + 3\pi^2\sigma^2) - 2v_0(H + 4\pi^2(\phi - 1))(\pi^2\sigma^2 - 2\Lambda v_0^4)) + 2Hv_0v_2(2\Lambda v_0^4 - \pi^2\sigma^2)) - \delta_0^2v_0^2(Hv_0^2 + \pi^2(H(5 - 4\phi) + 16\pi^2(\phi - 1)))(2\Lambda v_0^2 + \sigma^2)) + F_1(H(\pi^2\Lambda\mu_1^4(v_1(3Hv_1 - 2v_0(H + 4\pi^2(\phi - 1))) - 2Hv_0v_2) + v_0^4(v_1(2v_0(H + 4\pi^2(\phi - 1))(2\Lambda v_0^2 + \pi^2\Lambda + \sigma^2) + Hv_1(6\Lambda v_0^2 + \pi^2\Lambda + \sigma^2)) + 2Hv_0v_2(2\Lambda v_0^2 + \pi^2\Lambda + \sigma^2))) - \delta_0^2v_0^2(Hv_0^2 + \pi^2(H(5 - 4\phi) + 16\pi^2(\phi - 1)))(\Lambda(\mu_1^4 + v_0^4) + v_0^2\sigma^2)) + 32\pi^4F_3^2v_0^4Ra_{nl}(\phi - 1)) \right). \quad (159)$$

Asymptotic Analysis for Large Values of γH

For large values of γH with Ra_{nl}^{RR} given by Eq. (88) and ν given by Eq. (90), Nu_w^{RR} takes the form:

$$Nu_w^{RR} = G_{22} + G_{23} \frac{1}{\gamma H} + G_{24} \frac{1}{\gamma H^2} + \cdots, \quad (160)$$

where

$$G_{22} = 1 + \frac{2\pi\phi(2F_3^2v_0^2Ra_{nl} - \delta_0^2Q_1)}{F_3F_4v_0^2Ra_{nl}}, \quad (161)$$

$$G_{23} = -\frac{1}{F_3F_4v_0^3Ra_{nl}} \left(2\pi(F_2\mu_1^2(v_0(2\Lambda v_0^2 + \sigma^2)(\delta_0^2H + 4\pi^2\delta_0^2(\phi - 1)) - 2v_1(\pi^2\sigma^2 - 2\Lambda v_0^4)) + F_1(v_0(\delta_0^2H + 4\pi^2\delta_0^2(\phi - 1))(\Lambda(\mu_1^4 + v_0^4) + v_0^2\sigma^2) + 2v_1(-\pi^2\Lambda\mu_1^4 + v_0^4(\pi^2\Lambda + \sigma^2) + 2\Lambda v_0^6)) - 8\pi^2F_3^2v_0^3Ra_{nl}(\phi - 1)) \right), \quad (162)$$

$$G_{24} = -\frac{1}{F_3F_4v_0^4Ra_{nl}} \left(2\pi(F_2\mu_1^2(2v_0^4(\Lambda(4\pi^2\delta_0^2H\phi + v_1^2) - H(5\pi^4\Lambda + 3\pi^2\sigma^2) - 8\pi^4(\phi - 1)(2\pi^2\Lambda + \sigma^2)) + \pi^2v_0^2\sigma^2(4\delta_0^2H\phi - 5\pi^2H - 16\pi^4(\phi - 1)) - v_0^6(H(12\pi^2\Lambda + \sigma^2) + 32\pi^4\Lambda(\phi - 1)) + 4\Lambda v_0^5(v_1(H + 4\pi^2(\phi - 1)) + v_2) - 2H\Lambda v_0^8 - 2\pi^2v_0\sigma^2(v_1(H + 4\pi^2(\phi - 1)) + v_2) + 3\pi^2v_1^2\sigma^2) + F_1(Hv_0(2v_1(v_0^4(2\Lambda v_0^2 + \pi^2\Lambda + \sigma^2) - \pi^2\Lambda\mu_1^4) - v_0(-4\pi^2\delta_0^2\phi + v_0^4 + 6\pi^2v_0^2 + 5\pi^4))(\Lambda(\mu_1^4 + v_0^4) + v_0^2\sigma^2)) + v_0^4(v_1^2$$

$$\begin{aligned}
& (\pi^2 \Lambda + \sigma^2) - 16\pi^4 (\phi - 1) (\Lambda \mu_1^4 + \pi^2 \sigma^2)) - 16\pi^6 \Lambda \mu_1^4 v_0^2 (\phi - 1) \\
& - 2\pi^2 \Lambda \mu_1^4 v_0 (4\pi^2 v_1 (\phi - 1) + v_2) + 3\pi^2 \Lambda \mu_1^4 v_1^2 + 2v_0^6 (3\Lambda v_1^2 - 8\pi^4 \\
& (\phi - 1) (\pi^2 \Lambda + \sigma^2)) + 2v_0^5 (\pi^2 \Lambda + \sigma^2) (4\pi^2 v_1 (\phi - 1) + v_2) \\
& - 16\pi^4 \Lambda v_0^8 (\phi - 1) + 4\Lambda v_0^7 (4\pi^2 v_1 (\phi - 1) + v_2)) \\
& + 32\pi^4 F_3^2 v_0^4 Ra_{nl} (\phi - 1))
\end{aligned} \tag{163}$$

In the next section we discuss the results obtained from linear and non-linear stability analyses and make a few conclusions.

Results and Discussion

Linear and weakly nonlinear stability analyses of Rayleigh–Bénard convection for free–free and rigid–rigid, isothermal boundaries using LTNE model is studied. The thermophysical properties of ethylene-glycol [17] as the base liquid, copper [17] as the nanoparticle and 30% glass fiber reinforced polycarbonate porous material [8] are considered. The thermophysical properties of porous material and ethylene-glycol copper saturated porous medium is calculated and the same is tabulated in Tables 1 and 2.

The effects of LTNE is found to be important only when $H \gg 0$ and $\gamma \gg 0$. The definitions of H and γ both contain ϕ which makes it clear that when ϕ is close to 1 we have $\gamma H \gg 1$ and when ϕ is close to 0 then H is large. Table 11 gives us the exact meaning of $H \rightarrow 0$ and $H \rightarrow \infty$ for large values of γ , for example, in the case of free–free boundaries with a value of 10 for γ , we see that $H \rightarrow 0$ would mean any value of H equal to or smaller than 0.057 and the same is presented in Fig. 2. Similarly, $H \rightarrow \infty$ would mean any value of H equal to or greater than 218.53, and this is presented in Fig. 3. From Table 11, it is clear that the exact meaning of $H \rightarrow 0$ and $H \rightarrow \infty$ predominantly depends on the value of γ and less predominantly on the type of velocity boundary condition. In addition, Table 12 reveals the exact meaning of $\gamma \rightarrow \infty$ and $\gamma H \rightarrow \infty$ for large values of H . For example in the case of rigid–rigid boundaries with a value of 10 for H , we see that $\gamma \rightarrow \infty$ would mean any value of γ equal to or greater than 229.3 and the same is presented in Fig. 4. Similarly, $\gamma H \rightarrow \infty$ would mean any value of γH equal to or greater than 2292.98, and this is presented in Fig. 5. The comparison of exact and asymptotic solutions for small values of H , large values of H , γ and γH in case of free–free and rigid–rigid boundaries is demonstrated in Tables 3, 4, 5, 6, 7, 8, 9 and 10 and are in good agreement with each other as is to be expected.

From Figs. 6 and 7, it is clear that the effect of increasing H on the onset of convection is to stabilize the system. The value of the critical Rayleigh number of LTNE matches with that of LTE for large values of H . Thus the classical results of LTE case are obtained as a limiting case of LTNE for large values of H . The critical Rayleigh number of LTNE case is given by

$$Ra_{nl}^{LTE} = \left(\frac{\gamma}{1 + \gamma} \right) Ra_{nl}^{LTNE}. \tag{164}$$

From Figs. 8 and 9, it is apparent that the effect of increasing γ on the onset of convection is to destabilize the system and it is significant for small values of γ . Hence H and γ have opposing influences on the onset of convection (Tables 11, 12).

We now discuss the non-linear stability results:

From Figs. 6 and 7 we observe the effect of H on heat transport. On increasing the value of H , weighted average Nusselt number decreases due to increase in rapid heat exchange

between the phases and increase in residence time of heat in solid phase. Hence the effect of H is to diminish the amount of heat transport.

The effect of increasing the value of γ enhances the amount of heat transport due to high thermal conductivity of the nanoliquid and low thermal conductivity of the porous medium which is shown in Figs. 8 and 9. Thus H and γ have opposing effects on the heat transport.

The critical Rayleigh number increases with increase in Λ and this result is depicted in Figs. 10 and 11. Increase in the value of Λ signifies decrease in the value of permeability and hence less space is available for the nanoliquid to flow. Hence this results in delayed onset of convection and decrease in heat transport.

The critical Rayleigh number increases with increase in σ^2 and this is shown in Figs. 12 and 13. The Nusselt number decreases with increase in σ^2 which is also depicted in Figs. 12 and 13. The low thermal conductivity of the porous medium slows down the flow and diminishes the heat transport.

From Figs. 14 and 15, it is clear that the onset of convection is advanced in the case of LTNE compared to that in the case of LTE for both free–free and rigid–rigid boundaries. Also, the amount of heat transport is more in the case of LTNE than in the case of LTE.

The onset of convection is delayed in the case of rigid–rigid boundaries than in the case free–free boundaries and this result is depicted in Fig. 16.

In the present study, the value of critical Rayleigh number is large due to the presence of porous medium which has a very low thermal conductivity and retains the heat. Experimental investigation of the above study has been carried out by Katto and Masuoka [9].

Conclusion

- (a) The critical value of Rayleigh number and Nusselt number of the nanoliquid saturated porous medium vary as shown below.

$$Ra_{nlc}^{FF} < Ra_{nlc}^{FF}, Nu_w^{FF} > Nu_w^{RR}$$

- (b) The effect of LTNE ceases both in the presence and absence of rotation for small values of H , large values of H , γ and γH .

- (c) $Ra_{nlc}^{LTNE} < Ra_{nlc}^{LTE}$, $Nu_w^{LTNE} > Nu_w^{LTE}$

- (d) The effect of increasing Λ , σ^2 and H is to stabilize the system whereas the effect of increasing γ is to destabilize the system.

$$(i) \frac{d}{d\Lambda} (Ra_{nlc}) > 0, \frac{d}{d\Lambda} (Nu_w) < 0.$$

$$(ii) \frac{d}{d\sigma^2} (Ra_{nlc}) > 0, \frac{d}{d\sigma^2} (Nu_w) < 0.$$

$$(iii) \frac{d}{dH} (Ra_{nlc}) > 0, \frac{d}{dH} (Nu_w) < 0.$$

$$(iv) \frac{d}{d\gamma} (Ra_{nlc}) < 0, \frac{d}{d\gamma} (Nu_w) > 0.$$

- (e) The onset of convection is delayed in the case of rigidrigid, isothermal boundaries than that in the case of freefree, isothermal boundaries which in turn leads to a diminished heat transport situation in the former case compared to the latter.

- (f) Asymptotic and Galerkin solutions are found to be in good agreement with each other.

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