PBIB-designs and association schemes arising from minimum bi-connected dominating sets of some special classes of graphs

B. Chaluvvaraju¹ · N. Manjunath¹

Received: 12 July 2016 / Accepted: 21 August 2017 / Published online: 8 September 2017
© African Mathematical Union and Springer-Verlag GmbH Deutschland 2017

Abstract A dominating set D of a connected graph $G = (V, E)$ is said to be bi-connected dominating set if the induced subgraphs of both $\langle D \rangle$ and $\langle V - D \rangle$ are connected. The bi-connected domination number $\gamma_{bc}(G)$ is the minimum cardinality of a bi-connected dominating set. A γ_{bc} -set is a minimum bi-connected dominating set of G . In this paper, we obtain the Partially Balanced Incomplete Block (PBIB)-designs with $m = 1, 2, 3, 4$ and $\lfloor \frac{p}{2} \rfloor$ association schemes arising from γ_{bc} -sets of some special classes of graphs.

Keywords Association schemes · Partially balanced incomplete block designs · Bi-connected dominating sets · Strongly regular graph · Circulant graph · Hypercube

Mathematics Subject Classification Primary 05C51 · 05E30 · 05C69

1 Introduction

Throughout this paper, the graphs chosen are finite, undirected, simple and connected. A graph $G = (V, E)$ with $p = |V|$ and $q = |E|$ denote the number of vertices and edges, respectively. In general we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X . $N(v)$ and $N[v]$ denote open and closed neighborhoods of a vertex v respectively. For graph-theoretical terminology and notation not defined here we follow [6].

Bose and Nair [2] introduced a class of binary, equireplicate and proper designs, which are called partially balanced incomplete block (PBIB)-designs. This design is classified into different types on the basis of their association schemes. In brief, they can be grouped as PBIB(2)-designs, PBIB(3)-designs and higher associate-class PBIB-designs. In each of the

✉ B. Chaluvvaraju
bchaluvvaraju@gmail.com

N. Manjunath
manjubub@gmail.com

¹ Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bangalore 560 056, India

above mentioned groups there are further sub groups based on the types of the underlying association scheme.

Given ν objects (elements or vertices) a relation satisfying the following conditions is said to be an association scheme with m classes:

- (i) Any two objects are either first associates, or second associates, $\dots$, or m th associates, the relation of association being symmetric.
- (ii) Each object α has n_i i th associates, the number n_i being independent of α .
- (iii) If two objects α and β are k th associates, then the number of objects which are i th associates of α and j th associates of β is p_{ij}^k and is independent of the k th associates α and β . Also $p_{ij}^k = p_{ji}^k$.

With the association scheme on ν objects, the PBIB-design is defined as follows.

The PBIB-designs is an arrangement of ν objects into b sets of size s where $s < \nu$ such that

- (i) Every object is contained in exactly r blocks.
- (ii) Each block contains s distinct objects.
- (iii) Any two objects which are k th associates occur together in exactly λ_k blocks.

The numbers $\nu, b, r, s, \lambda_1, \lambda_2, \dots, \lambda_m$ are called the parameters of the first kind, whereas the numbers $n_1, n_2, \dots, n_m, p_{ij}^k$ ($i, j, k = 1, 2, \dots, m$) are called the parameters of the second kind.

The concept of domination has existed and has been studied from a long time. Books on domination [7] has stimulated sufficient inspiration, leading to the expansive growth of this field. A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G . Further, a dominating set D is a connected dominating set, if the induced subgraph $\langle D \rangle$ is connected, we refer to [8, 12]/nonsplit (or, co-connected) dominating set, if the induced subgraph $\langle V - D \rangle$ is connected, we refer the reader [3, 9]/doubly connected (or, bi-connected) dominating set, if the induced subgraphs of both $\langle D \rangle$ and $\langle V - D \rangle$ are connected, we refer to [4]. The minimum cardinality taken over all connected/co-connected/bi-connected dominating set in G is called a connected/co-connected/bi-connected domination number of G and is denoted by $\gamma_c(G)/\gamma_{cc}(G)/\gamma_{bc}(G)$, respectively. A γ_{bc} -set is a minimum bi-connected dominating set of G .

In [14], Walikar et al. have studied on the number of minimum dominating sets and PBIB-designs arising from minimum dominating sets of paths and cycles. PBIB-designs and association schemes arising from minimum connected and co-connected dominating sets of some special kinds of graphs were initiated by us in [10]. Analogously, in this paper, we obtain the PBIB-designs with $m = 1, 2, 3, 4$ and $\lfloor \frac{p}{2} \rfloor$ association schemes arising from γ_{bc} -sets of some special classes of graphs.

2 Trees with non-existence of association schemes

A non trivial Tree T is a connected acyclic simple graph with $p \geq 2$ vertices. For non-existence of PBIB-design and associate classes in certain families of graphs, we follow [13].

Theorem 2.1 *The collection of all γ_{bc} -sets of a tree T of order at least 3 does not form a PBIB-design.*

Proof Let T be any tree on $p \geq 3$ vertices. Clearly, every γ_{bc} -sets contain $p - 1$ vertices and there exists a number of γ_{bc} -sets equal to the number of pendant vertices, say ρ in T . Every pendant vertex is repeated in $\rho - 1$ number of γ_{bc} -sets, but every non-pendant vertex is repeated in ρ times of γ_{bc} -sets. Hence, the collection of all γ_{bc} -sets does not form a PBIB-design. $\square$

3 Complete graphs with $m = 1$ association scheme

A complete graph K_p with $p \geq 2$ vertices is a graph in which every pair of vertices is adjacent.

Theorem 3.1 *The collection of all γ_{bc} -sets of K_p ; $p \geq 3$ forms a PBIB-design with 1-association scheme and parameters $v = p$, $b = p$, $r = 1$, $s = 1$ and $\lambda_1 = 0$.*

Proof Let a complete graph K_p be labeled as $v_1, v_2, \dots, v_p$ with the collection of all γ_{bc} -sets are given by $\{v_i\}$, $1 \leq i \leq p$. Then two vertices u and v are 1st associates if they are adjacent. For each vertex $v_i \in V(K_p)$, the vertices v_j , $1 \leq j \neq i \leq p$ are the 1st associates. The parameters of second kind are given by $n_1 = p - 1$ and $P^1 = [p_{11}^1] = [p - 2]$ and the parameters of first kind are given by $v = p$, $b = p$, $r = 1$, $s = 1$ and $\lambda_1 = 0$. $\square$

4 Strongly regular graph with $m = 2$ association scheme

A simple graph of order p is a strongly regular graph with parameters (p, l, λ, μ) whenever it is not complete or edgeless and

- (i) each vertex is adjacent to l vertices,
- (ii) for each pair of adjacent vertices there are λ vertices adjacent to both,
- (iii) for each pair of non-adjacent vertices there are μ vertices adjacent to both.

For example, the cycle C_5 is a strongly regular graph having parameters $(p, l, \lambda, \mu) = (5, 2, 0, 1)$.

Remark 1 The complete graph K_p is strongly regular for all $p \geq 3$. The status of the trivial singleton graph K_1 is unclear. Opinions differ on if K_2 is a strongly regular graph, since it has no well-defined μ parameter, it is preferable to consider as not to be a strongly regular.

For more details on strongly regular graph and its related concepts, we refer to [1, 5].

Theorem 4.1 *The set of all γ_{bc} -set in the strongly regular graph G with the parameters (p, l, λ, μ) , where v any vertex of G and $\mu \geq 1$ are blocks of PBIB-design with the parameters, $v = p$, $b = p$, $r = l$, $s = l$, $n_1 = l$ and $n_2 = p - l - 1$, with*

$$P^1 = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} \lambda & l - \lambda - 1 \\ l - \lambda - 1 & p - 2l + \lambda \end{pmatrix} \text{ and}$$

$$P^2 = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix} = \begin{pmatrix} \mu & l - \mu \\ l - \mu & p - 2l + \mu - 2 \end{pmatrix}.$$

Proof Let G be a strongly regular graph with the parameters (p, l, λ, μ) and let v be any vertex of G , we can define the PBIB-design by taking the vertices of G as the vertex set, the

block set as γ_{bc} -set, and for any two elements α and β , they are first associates if the vertices are adjacent in G and second associates if the vertices are not adjacent in G . It is easy to see that every vertex (element) of the strongly regular graph belong to exactly $(l + 1)$ -blocks, which is a γ_{bc} -set, so the parameters of the PBIB-designs are $v, b, r, k, \lambda_1, \lambda_2, n_1$ and n_2 , where $v = b = p, r = k = l, \lambda_1 = 0, \lambda_2 = \mu, n_1 = l$ and $n_2 = p - l - 1$. Further, from the definition of the strongly regular graph, we have $p_{11}^1 = \lambda$ and $p_{11}^2 = \mu$. Also it is well known that, $p_{11}^1 + p_{12}^1 + 1 = p_{11}^2 + p_{12}^2 = n_1$ and $p_{12}^1 + p_{22}^1 = p_{21}^2 + p_{22}^2 + 1 = n_2$. Thus the result follows. $\square$

Now, we show some strongly regular graph with $m = 2$ -associate classes.

4.1 Petersen graph with 2-association scheme

The Petersen graph P is a cubic symmetric graph and is nonplanar with the order 10. A danish Mathematician Julius Petersen [11] constructed a unique strongly regular graph with parameters $p = 10, l = 3, \lambda = 0$ and $\mu = 1$.

Theorem 4.2 *The collection of all γ_{bc} -sets of Petersen graph P forms a PBIB-design with 2-association scheme and parameters $v = 10, b = 10, r = 4, s = 4, \lambda_1 = 2$ and $\lambda_2 = 1$.*

Proof Let a Petersen graph P be labeled as $v_1, v_2, \dots, v_{10}$, which is a strongly regular graph with the parameters $(10, 3, 0, 1)$ as shown in Fig. 1.

The collection of all γ_{bc} -sets is given by:

$$\begin{aligned} & \{v_1, v_2, v_6, v_{10}\}, \{v_1, v_2, v_3, v_9\}, \{v_2, v_3, v_4, v_7\}, \{v_3, v_4, v_5, v_{10}\}, \\ & \{v_4, v_5, v_6, v_9\}, \{v_1, v_5, v_6, v_7\}, \{v_3, v_6, v_7, v_8\}, \{v_7, v_8, v_9, v_{10}\}, \\ & \{v_2, v_5, v_8, v_9\}, \{v_1, v_4, v_8, v_{10}\}. \end{aligned}$$

By considering the γ_{bc} -sets above as blocks, the two association scheme can be defined as, two elements (vertices) α and β are first associates if the vertices are adjacent in G and they are second associates, otherwise. The first and second associates of the elements of Petersen graph P are given in Table 1.

Fig. 1 Petersen graph

Table 1 1st and 2nd associates of the vertices of Petersen graph

Elements	First associates	Second associates
v_1	v_2, v_6, v_{10}	$v_3, v_4, v_5, v_7, v_8, v_9$
v_2	v_1, v_3, v_9	$v_4, v_5, v_6, v_7, v_8, v_{10}$
v_3	v_2, v_4, v_7	$v_1, v_5, v_6, v_8, v_9, v_{10}$
v_4	v_3, v_5, v_{10}	$v_1, v_2, v_6, v_7, v_8, v_9$
v_5	v_4, v_6, v_9	$v_1, v_2, v_3, v_7, v_8, v_{10}$
v_6	v_1, v_5, v_7	$v_2, v_3, v_4, v_8, v_9, v_{10}$
v_7	v_3, v_6, v_8	$v_1, v_2, v_4, v_5, v_9, v_{10}$
v_8	v_7, v_9, v_{10}	$v_1, v_2, v_3, v_4, v_5, v_6$
v_9	v_2, v_5, v_8	$v_1, v_3, v_4, v_6, v_7, v_{10}$
v_{10}	v_1, v_4, v_8	$v_2, v_3, v_5, v_6, v_7, v_9$

With this association, the parameters of second kind are given by $n_1 = 3$ and $n_2 = 6$ with

$$P^1 = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} \text{ and}$$

$$P^2 = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

Thus, the parameters of first kind are given by $v = 10$, $b = 10$, $r = 4$, $s = 4$, $\lambda_1 = 2$ and $\lambda_2 = 1$. $\square$

4.2 Clebsch graph with 2-association scheme

Clebsch graph or a Qunitic graph is a 5-regular graph and occurs only on even number of vertices, named after a German mathematician Alfred Clebsch, this graph has parameters $p = 16$, $l = 5$, $\lambda = 0$, $\mu = 2$. The subgraphs which are induced by the ten non-adjacent vertices in a Clebsch graph form an isomorphic copy of the Petersen graph. For more details, refer [5].

Theorem 4.3 *The collection of all γ_{bc} -sets of Clebsch graph C forms a PBIB-design with 2-association scheme and parameters $v = 16$, $b = 16$, $r = 6$, $s = 6$, $\lambda_1 = 2$ and $\lambda_2 = 2$.*

Proof Let a Clebsch graph C be labeled as $v_1, v_2, \dots, v_{16}$, which is a strongly regular graph with the parameters $(16, 5, 0, 2)$ as shown in Fig. 2.

The collection of all γ_{bc} -sets is given by:

$$\begin{aligned} &\{v_1, v_2, v_7, v_{10}, v_{14}, v_{16}\}, \{v_1, v_2, v_3, v_5, v_9, v_{12}\}, \{v_2, v_3, v_4, v_7, v_{13}, v_{15}\}, \\ &\{v_3, v_4, v_5, v_8, v_{10}, v_{16}\}, \{v_2, v_4, v_5, v_6, v_{11}, v_{14}\}, \{v_5, v_6, v_7, v_9, v_{13}, v_{16}\}, \\ &\{v_1, v_3, v_6, v_7, v_8, v_{11}\}, \{v_4, v_7, v_8, v_9, v_{12}, v_{14}\}, \{v_2, v_6, v_8, v_9, v_{10}, v_{15}\}, \\ &\{v_1, v_4, v_9, v_{10}, v_{11}, v_{13}\}, \{v_5, v_7, v_{10}, v_{11}, v_{12}, v_{15}\}, \{v_2, v_8, v_{11}, v_{12}, v_{13}, v_{16}\}, \\ &\{v_3, v_6, v_{10}, v_{12}, v_{13}, v_{14}\}, \{v_1, v_5, v_8, v_{13}, v_{14}, v_{15}\}, \{v_3, v_9, v_{11}, v_{14}, v_{15}, v_{16}\}, \\ &\{v_1, v_4, v_6, v_{12}, v_{15}, v_{16}\}. \end{aligned}$$

By considering the γ_{bc} -sets above as blocks, the two association scheme can be defined as, two elements (vertices) α and β are first associates if the vertices are adjacent in G and

Fig. 2 Clebsch graph **Table 2** 1st and 2nd associates of the vertices of Clebsch graph

Elements	First associates	Second associates
v_1	$v_2, v_7, v_{10}, v_{14}, v_{16}$	$v_3, v_4, v_5, v_6, v_8, v_9, v_{11}, v_{12}, v_{13}, v_{15}$
v_2	$v_1, v_3, v_5, v_9, v_{12}$	$v_4, v_6, v_7, v_8, v_{10}, v_{11}, v_{13}, v_{14}, v_{15}, v_{16}$
v_3	$v_2, v_4, v_7, v_{13}, v_{15}$	$v_1, v_5, v_6, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{14}, v_{16}$
v_4	$v_3, v_5, v_8, v_{10}, v_{16}$	$v_1, v_5, v_6, v_7, v_9, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}$
v_5	$v_2, v_4, v_6, v_{11}, v_{14}$	$v_1, v_3, v_7, v_8, v_9, v_{10}, v_{12}, v_{13}, v_{15}, v_{16}$
v_6	$v_5, v_7, v_9, v_{13}, v_{16}$	$v_1, v_2, v_3, v_4, v_6, v_8, v_{10}, v_{11}, v_{12}, v_{15}$
v_7	$v_1, v_3, v_6, v_8, v_{11}$	$v_2, v_4, v_5, v_9, v_{10}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}$
v_8	$v_4, v_7, v_9, v_{12}, v_{14}$	$v_1, v_2, v_3, v_5, v_6, v_{10}, v_{11}, v_{13}, v_{15}, v_{16}$
v_9	$v_2, v_6, v_8, v_{10}, v_{15}$	$v_1, v_3, v_4, v_5, v_7, v_{11}, v_{12}, v_{13}, v_{14}, v_{16}$
v_{10}	$v_1, v_4, v_9, v_{11}, v_{13}$	$v_2, v_3, v_5, v_6, v_7, v_8, v_{12}, v_{14}, v_{15}, v_{16}$
v_{11}	$v_5, v_7, v_{10}, v_{12}, v_{15}$	$v_1, v_2, v_3, v_4, v_6, v_8, v_9, v_{13}, v_{14}, v_{16}$
v_{12}	$v_2, v_8, v_{11}, v_{13}, v_{16}$	$v_1, v_3, v_4, v_5, v_6, v_7, v_9, v_{10}, v_{14}, v_{15}$
v_{13}	$v_3, v_6, v_{10}, v_{12}, v_{14}$	$v_1, v_2, v_4, v_5, v_7, v_8, v_9, v_{11}, v_{14}, v_{16}$
v_{14}	$v_1, v_5, v_8, v_{13}, v_{15}$	$v_2, v_4, v_6, v_7, v_9, v_{10}, v_{11}, v_{12}, v_{15}, v_{16}$
v_{15}	$v_3, v_9, v_{11}, v_{14}, v_{16}$	$v_1, v_2, v_4, v_5, v_6, v_7, v_8, v_{10}, v_{12}, v_{13}$
v_{16}	$v_1, v_4, v_6, v_{12}, v_{15}$	$v_2, v_3, v_5, v_7, v_8, v_9, v_{10}, v_{11}, v_{13}, v_{14}$

they are second associates if the vertices are not adjacent. The first and second associates of the vertices of Clebsch graph are as given in Table 2.

With this association, parameters of second kind are $n_1 = 6$ and $n_2 = 10$ with

$$P^1 = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 6 \end{pmatrix} \text{ and}$$

$$P^2 = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}.$$

Thus, the parameters of first kind are given by $v = 16$, $b = 16$, $s = 6$, $r = 6$, $\lambda_1 = 2$ and $\lambda_2 = 2$. $\square$

5 Circulant graphs with $m = 2$ association schemes

Circulant graphs are considered in the design of computer and telecommunication networks, their popularity is due to their optimal fault-tolerance and routing capabilities.

For a given positive integer p , let $s_1, s_2, \dots, s_k$ be a sequence of integers where $0 < s_1 < s_2 < \dots < s_k < \frac{p+1}{2}$. Then the Circulant graph $C_p(s_1, s_2, \dots, s_k)$ is the graph on p vertices $v_1, v_2, \dots, v_p$ with vertex v_i adjacent to each vertex $v_{i \pm s_j \pmod p}$. The values s_i are called *jump sizes*.

Theorem 5.1 *The collection of all γ_{bc} -sets of $C_p(1, 2, \dots, \frac{p-2}{2})$ form PBIB-design with 2-association scheme and parameters $v = p$, $b = \frac{p(p-2)}{2}$, $r = p - 2$, $s = 2$, $\lambda_1 = 1$ and $\lambda_2 = 0$, where p is even.*

Proof Let a circulant graph $C_p(1, 2, \dots, \frac{p-2}{2})$ be labeled as $v_1, v_2, \dots, v_p$, as shown in Fig. 3.

The set of all γ_{bc} -sets are given as follows:

For $1 \leq i \leq \frac{p}{2}$, $A_i = \{\{v_i, v_j\} : v_j \in G, j \neq k, 1 \leq k \leq i, j \neq \frac{p}{2} + i\}$.

For $\frac{p}{2} < i \leq p - 1$, $A_i = \{\{v_i, v_j\} : v_j \in G, j \neq k, 1 \leq k \leq i\}$.

Then two vertices u and v are said to be first associated if $d(u, v) = 1$ and second associated if $d(u, v) = 2$. The first and second associates of each elements are shown in the Table 3.

With this association, the parameters of second kind are given by $n_1 = p - 2$ and $n_2 = 1$ with

$$P^1 = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} p-4 & 1 \\ 1 & 0 \end{pmatrix} \text{ and}$$

$$P^2 = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix} = \begin{pmatrix} p-2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the parameters of first kind are given by $v = p$, $b = \frac{p(p-2)}{2}$, $r = p - 2$, $s = 2$, $\lambda_1 = 1$, $\lambda_2 = 0$. $\square$

For illustration, we consider the collection of all γ_{bc} -sets of a circulant graph $C_8(1, 2, 3)$, which form a PBIB-design with 2-association scheme and parameters $v = 8$, $b = 24$, $r = 6$, $s = 2$, $\lambda_1 = 1$ and $\lambda_2 = 0$.

Fig. 3 Circulant graph $C_p(1, 2, \dots, \frac{p-2}{2})$

Table 3 1st and 2nd associates of circulant graph $C_p(1, 2, \dots, \frac{p-2}{2})$

Elements	First associates	Second associates
v_1	$\{v_j \in G : j \neq 1 \text{ and } j \neq \frac{p}{2} + 1\}$	$v_{\frac{p}{2}+1}$
v_2	$\{v_j \in G : j \neq 2 \text{ and } j \neq \frac{p}{2} + 2\}$	$v_{\frac{p}{2}+2}$
$\vdots$	$\vdots$	$\vdots$
$v_{\frac{p}{2}}$	$\{v_j \in G : j \neq \frac{p}{2} \text{ and } j \neq p\}$	v_p
$v_{\frac{p}{2}+1}$	$\{v_j \in G : j \neq \frac{p}{2} + 1 \text{ and } j \neq (p+1)(\text{mod } p)\}$	v_1
$v_{\frac{p}{2}+2}$	$\{v_j \in G : j \neq \frac{p}{2} + 2 \text{ and } j \neq (p+2)(\text{mod } p)\}$	v_2
$\vdots$	$\vdots$	$\vdots$
$v_{\frac{p}{2}+i}$	$\{v_j \in G : j \neq \frac{p}{2} + i \text{ and } j \neq (p+i)(\text{mod } p)\}$	v_i
$\vdots$	$\vdots$	$\vdots$
v_p	$\{v_j \in G : j \neq p \text{ and } j \neq (\frac{p}{2} + p)(\text{mod } p)\}$	$v_{\frac{p}{2}}$

Fig. 4 Circulant graph $C_8(1, 2, 3)$

Let a circulant graph $C_8(1, 2, 3)$ be labeled as $v_1, v_2, \dots, v_8$, as shown in the figure below (Fig. 4):

The set of all γ_{bc} -sets are given as follows:

For $1 \leq i \leq \frac{p}{2} = 4$,

$$A_i = \left\{ \{v_i, v_j\} : v_j \in G, j \neq i, 1 \leq k \leq i, j \neq \frac{p}{2} + i \right\}.$$

If $i = 1$, then $A_1 = \{\{v_1, v_j\} : v_j \in G, j \neq 1, 1 \leq k \leq 1, j \neq 4 + 1\}$.

That is, $A_1 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_6\}, \{v_1, v_7\}, \{v_1, v_8\}\}$.

Similarly,

Table 4 1st and 2nd associates of circulant graph $C_8(1, 2, 3)$

Elements	First associates	Second associates
v_1	$\{v_j \in G : j \neq 1 \text{ and } j \neq 5\}$	v_5
v_2	$\{v_j \in G : j \neq 2 \text{ and } j \neq 6\}$	v_6
v_3	$\{v_j \in G : j \neq 3 \text{ and } j \neq 7\}$	v_7
v_4	$\{v_j \in G : j \neq 4 \text{ and } j \neq 8\}$	v_8
v_5	$\{v_j \in G : j \neq 5 \text{ and } j \neq 1\}$	v_1
v_6	$\{v_j \in G : j \neq 6 \text{ and } j \neq 2\}$	v_2
v_7	$\{v_j \in G : j \neq 7 \text{ and } j \neq 3\}$	v_3
v_8	$\{v_j \in G : j \neq 8 \text{ and } j \neq 4\}$	v_4

If $i = 2$, then $A_2 = \{\{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_7\}, \{v_2, v_8\}\}$.

If $i = 3$, then $A_3 = \{\{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_3, v_8\}\}$.

If $i = 4$, then $A_4 = \{\{v_4, v_5\}, \{v_4, v_6\}, \{v_4, v_7\}\}$.

Now, for $\frac{p}{2} = 4 < i \leq p - 1 = 7$,

$$A_i = \{\{v_i, v_j\} : v_j \in G, j \neq k, 1 \leq k \leq i\}.$$

If $i = 5$, then $A_5 = \{\{v_5, v_j\} : v_j \in G, j \neq k, 1 \leq k \leq 5\}$.

That is, $A_5 = \{\{v_5, v_6\}, \{v_5, v_7\}, \{v_5, v_8\}\}$.

Similarly,

If $i = 6$, then $A_6 = \{\{v_6, v_7\}, \{v_6, v_8\}\}$.

If $i = 7$, then $A_7 = \{\{v_7, v_8\}\}$.

Then the two vertices u and v are said to be first associated if $d(u, v) = 1$, second associated if $d(u, v) = 2$. The first and second associates of each elements are specified in Table 4.

With this association, the parameters of second kind are given by $n_1 = p - 2 = 6$ and $n_2 = 1$ with

$$P^1 = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} p-4 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \text{ and}$$

$$P^2 = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix} = \begin{pmatrix} p-2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the parameters of first kind are given by $v = p = 8$, $b = \frac{p(p-2)}{2} = 24$, $r = p - 2 = 6$, $s = 2$, $\lambda_1 = 1$ and $\lambda_2 = 0$.

6 Hypercubes with $m = 1, 2, 3$ and 4 association schemes

The n -dimensional cube or hypercube Q_n is the simple graph whose vertices are the n -tuples with entries in 0, 1 and whose edges are the pairs of n -tuples that differ in exactly one position. Let Q_0 be a hypercube of dimension zero. Then the order of Q_0 is one. Clearly, for γ_{bc} -sets we consider only hypercube Q_n with $n \geq 1$. The interest in hypercube has increased in the field of graph theory and has found its applications in the advent of massively parallel computers, whose structure results that of a hypercube.

Theorem 6.1 *The collection of all γ_{bc} -sets of a hypercube Q_1 forms a PBIB-design with 1-association scheme and parameters $v = 2, b = 2, r = 1, s = 1$ and $\lambda_1 = 0$.*

Proof By Theorem 3.1, complete graph K_2 is isomorphic with Q_1 . Thus the results follows. $\square$

Theorem 6.2 *The collection of all γ_{bc} -sets of a hypercube Q_2 forms a PBIB-design with 2-association scheme and parameters $v = 4, b = 4, r = 2, s = 2, \lambda_1 = 1$ and $\lambda_2 = 0$.*

Proof Let a hypercube Q_2 be labeled as $v_1, v_2, \dots, v_4$ with the collection of all γ_{bc} -sets are given by $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}$ and $\{v_4, v_1\}$. Then two vertices u and v are first associates if $d(u, v) = 1$ and second associates if there exists a vertex non-adjacent to any two of the adjacent vertices. Hence, the parameters of second kind are given by $n_1 = 2$. Thus the parameters of first kind are given by $v = 4, b = 4, r = 2, s = 2, \lambda_1 = 1$ and $\lambda_2 = 0$. $\square$

Theorem 6.3 *The collection of all γ_{bc} -sets of a hypercube Q_3 forms a PBIB-design with 3-association scheme and parameters $v = 8, b = 30, r = 15, s = 4, \lambda_1 = 8, \lambda_2 = 5$ and $\lambda_3 = 6$.*

Proof Let a hypercube Q_3 be labeled as $v_1, v_2, \dots, v_8$, as shown in Fig. 5.

The collection of all γ_{bc} -sets is given by

$$\begin{aligned} &\{v_1, v_2, v_6, v_7\}, \{v_2, v_3, v_7, v_8\}, \{v_3, v_4, v_8, v_5\}, \{v_4, v_1, v_5, v_6\}, \\ &\{v_5, v_6, v_7, v_8\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_6, v_5\}, \{v_2, v_3, v_7, v_6\}, \\ &\{v_3, v_4, v_8, v_7\}, \{v_4, v_1, v_5, v_8\}, \{v_1, v_6, v_7, v_8\}, \{v_2, v_7, v_8, v_5\}, \\ &\{v_3, v_8, v_5, v_6\}, \{v_4, v_5, v_6, v_7\}, \{v_1, v_2, v_3, v_6\}, \{v_2, v_3, v_4, v_7\}, \\ &\{v_3, v_4, v_1, v_8\}, \{v_4, v_1, v_2, v_5\}, \{v_1, v_6, v_5, v_8\}, \{v_2, v_7, v_6, v_5\}, \\ &\{v_3, v_8, v_7, v_6\}, \{v_4, v_5, v_8, v_7\}, \{v_1, v_2, v_7, v_8\}, \{v_2, v_3, v_8, v_5\}, \\ &\{v_3, v_4, v_5, v_6\}, \{v_4, v_1, v_6, v_7\}, \{v_1, v_2, v_3, v_8\}, \{v_2, v_3, v_4, v_5\}, \\ &\{v_3, v_4, v_1, v_6\}, \{v_4, v_1, v_2, v_7\}. \end{aligned}$$

Two distinct vertices u and v are said to be first associates if $d(u, v) = 1$; second associates if $d(u, v) = 2$ and third associates if $d(u, v) = 3$. The first, second and third associates of each vertices are shown in the Table 5.

Fig. 5 Hypercube Q_3

Table 5 1st, 2nd and 3rd associates of hypercube Q_3

Elements	First Associates	Second Associates	Third Associates
v_1	v_2, v_4, v_6	v_3, v_5, v_7	v_8
v_2	v_1, v_3, v_7	v_4, v_6, v_8	v_5
v_3	v_2, v_4, v_8	v_1, v_5, v_7	v_6
v_4	v_1, v_3, v_5	v_2, v_6, v_8	v_7
v_5	v_4, v_6, v_8	v_1, v_3, v_7	v_2
v_6	v_1, v_5, v_7	v_2, v_4, v_8	v_3
v_7	v_2, v_6, v_8	v_1, v_3, v_5	v_4
v_8	v_3, v_5, v_7	v_2, v_4, v_6	v_1

With this association, the parameters of second kind are given by $n_1 = 3, n_2 = 3$ and $n_3 = 1$ with

$$P^1 = \begin{pmatrix} p_{11}^1 & p_{12}^1 & p_{13}^1 \\ p_{21}^1 & p_{22}^1 & p_{23}^1 \\ p_{31}^1 & p_{32}^1 & p_{33}^1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$P^2 = \begin{pmatrix} p_{11}^2 & p_{12}^2 & p_{13}^2 \\ p_{21}^2 & p_{22}^2 & p_{23}^2 \\ p_{31}^2 & p_{32}^2 & p_{33}^2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and}$$

$$P^3 = \begin{pmatrix} p_{11}^3 & p_{12}^3 & p_{13}^3 \\ p_{21}^3 & p_{22}^3 & p_{23}^3 \\ p_{31}^3 & p_{32}^3 & p_{33}^3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the parameters of first kind are given by $v = 8, b = 30, r = 15, s = 4, \lambda_1 = 8, \lambda_2 = 5$ and $\lambda_3 = 6$. $\square$

Theorem 6.4 *The collection of all γ_{bc} -sets of a hypercube Q_4 form PBIB-design with 4-association scheme and parameters $v = 16, b = 196, r = 72, s = 6, \lambda_1 = 30, \lambda_2 = 16, \lambda_3 = 24$ and $\lambda_4 = 48$.*

Proof Let a hypercube Q_4 be labeled as $v_1, v_2, \dots, v_{16}$ as shown in Fig. 6.

The set of all γ_{bc} -sets are given by

$$\begin{aligned} &\{v_1, v_2, v_4, v_8, v_{11}, v_{12}\}, \{v_{13}, v_{14}, v_{16}, v_4, v_7, v_8\}, \{v_9, v_{10}, v_{12}, v_{16}, v_3, v_4\}, \\ &\{v_5, v_6, v_8, v_{12}, v_{15}, v_{16}\}, \{v_1, v_2, v_5, v_8, v_{11}, v_{12}\}, \{v_{13}, v_{14}, v_1, v_4, v_7, v_8\}, \\ &\{v_9, v_{10}, v_{13}, v_{16}, v_3, v_4\}, \{v_5, v_6, v_9, v_{12}, v_{15}, v_{16}\}, \{v_1, v_2, v_3, v_7, v_{11}, v_{12}\}, \\ &\{v_{13}, v_{14}, v_{15}, v_3, v_7, v_8\}, \{v_9, v_{10}, v_{11}, v_{15}, v_3, v_4\}, \{v_5, v_6, v_7, v_{11}, v_{15}, v_{16}\}, \\ &\{v_1, v_2, v_6, v_7, v_{11}, v_{12}\}, \{v_{13}, v_{14}, v_2, v_3, v_7, v_8\}, \{v_9, v_{10}, v_{14}, v_{15}, v_3, v_4\}, \\ &\{v_5, v_6, v_{10}, v_{11}, v_{15}, v_{16}\}, \{v_1, v_2, v_3, v_{11}, v_{12}, v_{15}\}, \{v_{13}, v_{14}, v_{15}, v_7, v_8, v_{11}\}, \\ &\{v_9, v_{10}, v_{11}, v_3, v_4, v_7\}, \{v_5, v_6, v_7, v_{15}, v_{16}, v_3\}, \{v_1, v_2, v_{11}, v_{12}, v_{14}, v_{15}\}, \\ &\{v_{13}, v_{14}, v_7, v_8, v_{10}, v_{11}\}, \{v_9, v_{10}, v_3, v_4, v_6, v_7\}, \{v_5, v_6, v_{15}, v_{16}, v_2, v_3\}, \\ &\{v_1, v_2, v_6, v_{10}, v_{11}, v_{12}\}, \{v_{13}, v_{14}, v_2, v_6, v_7, v_8\}, \{v_9, v_{10}, v_{14}, v_2, v_3, v_4\}, \\ &\{v_5, v_6, v_{10}, v_{14}, v_{15}, v_{16}\}, \{v_1, v_2, v_{10}, v_{11}, v_{12}, v_{14}\}, \{v_{13}, v_{14}, v_6, v_7, v_8, v_{10}\}, \\ &\{v_9, v_{10}, v_2, v_3, v_4, v_6\}, \{v_5, v_6, v_{14}, v_{15}, v_{16}, v_2\}, \{v_1, v_2, v_5, v_9, v_{11}, v_{12}\}, \end{aligned}$$

Fig. 6 Hypercube Q_4

$\{v_{13}, v_{14}, v_1, v_5, v_7, v_8\},$	$\{v_9, v_{10}, v_{13}, v_1, v_3, v_4\},$	$\{v_5, v_6, v_9, v_{13}, v_{15}, v_{16}\},$
$\{v_1, v_2, v_9, v_{11}, v_{12}, v_{13}\},$	$\{v_{13}, v_{14}, v_5, v_7, v_8, v_9\},$	$\{v_9, v_{10}, v_{14}, v_2, v_3, v_4\},$
$\{v_5, v_6, v_{10}, v_{14}, v_{15}, v_{16}\},$	$\{v_1, v_2, v_9, v_{11}, v_{12}, v_{13}\},$	$\{v_{13}, v_{14}, v_5, v_7, v_8, v_9\},$
$\{v_9, v_{10}, v_1, v_3, v_4, v_5\},$	$\{v_5, v_6, v_{13}, v_{15}, v_{16}, v_1\},$	$\{v_1, v_2, v_4, v_{11}, v_{12}, v_{16}\},$
$\{v_{13}, v_{14}, v_{16}, v_7, v_8, v_{12}\},$	$\{v_9, v_{10}, v_{12}, v_3, v_4, v_8\},$	$\{v_5, v_6, v_8, v_{15}, v_{16}, v_4\},$
$\{v_1, v_2, v_{11}, v_{12}, v_{13}, v_{16}\},$	$\{v_{13}, v_{14}, v_7, v_8, v_9, v_{12}\},$	$\{v_9, v_{10}, v_3, v_4, v_5, v_8\},$
$\{v_5, v_6, v_{15}, v_{16}, v_1, v_4\},$	$\{v_1, v_2, v_3, v_5, v_9, v_{12}\},$	$\{v_{13}, v_{14}, v_{15}, v_1, v_5, v_8\},$
$\{v_9, v_{10}, v_{11}, v_{13}, v_1, v_4\},$	$\{v_5, v_6, v_7, v_9, v_{13}, v_{16}\},$	$\{v_1, v_2, v_3, v_9, v_{12}, v_{13}\},$
$\{v_{13}, v_{14}, v_{15}, v_5, v_8, v_9\},$	$\{v_9, v_{10}, v_{11}, v_1, v_4, v_5\},$	$\{v_5, v_6, v_7, v_{13}, v_{16}, v_1\},$
$\{v_2, v_3, v_9, v_{12}, v_{13}, v_{14}\},$	$\{v_{14}, v_{15}, v_5, v_8, v_9, v_{10}\},$	$\{v_{10}, v_{11}, v_1, v_4, v_5, v_6\},$
$\{v_6, v_7, v_{13}, v_{16}, v_1, v_2\},$	$\{v_2, v_3, v_4, v_8, v_9, v_{12}\},$	$\{v_{14}, v_{15}, v_{16}, v_4, v_5, v_8\},$
$\{v_{10}, v_{11}, v_{12}, v_{16}, v_1, v_4\},$	$\{v_6, v_7, v_8, v_{12}, v_{13}, v_{16}\},$	$\{v_2, v_3, v_5, v_6, v_9, v_{12}\},$
$\{v_{14}, v_{15}, v_1, v_2, v_5, v_8\},$	$\{v_{10}, v_{11}, v_{13}, v_{14}, v_1, v_4\},$	$\{v_6, v_7, v_9, v_{10}, v_{13}, v_{16}\},$
$\{v_2, v_3, v_9, v_{10}, v_{12}, v_{14}\},$	$\{v_{14}, v_{15}, v_5, v_6, v_8, v_{10}\},$	$\{v_{10}, v_{11}, v_1, v_2, v_4, v_6\},$
$\{v_6, v_7, v_{13}, v_{14}, v_{16}, v_2\},$	$\{v_2, v_3, v_6, v_9, v_{10}, v_{12}\},$	$\{v_{14}, v_{15}, v_2, v_5, v_6, v_8\},$
$\{v_{10}, v_{11}, v_{14}, v_1, v_2, v_4\},$	$\{v_6, v_7, v_{10}, v_{13}, v_{14}, v_{16}\},$	$\{v_2, v_3, v_9, v_{11}, v_{12}, v_{15}\},$
$\{v_{14}, v_{15}, v_5, v_7, v_8, v_{11}\},$	$\{v_{10}, v_{11}, v_1, v_3, v_4, v_7\},$	$\{v_6, v_7, v_{13}, v_{15}, v_{16}, v_3\},$
$\{v_2, v_3, v_9, v_{12}, v_{15}, v_{16}\},$	$\{v_{14}, v_{15}, v_5, v_8, v_{11}, v_{12}\},$	$\{v_{10}, v_{11}, v_1, v_4, v_7, v_8\},$
$\{v_6, v_7, v_{13}, v_{16}, v_3, v_4\},$	$\{v_2, v_3, v_7, v_9, v_{11}, v_{12}\},$	$\{v_{14}, v_{15}, v_3, v_5, v_7, v_8\},$
$\{v_{10}, v_{11}, v_{15}, v_1, v_3, v_4\},$	$\{v_6, v_7, v_{11}, v_{13}, v_{15}, v_{16}\},$	$\{v_2, v_3, v_7, v_8, v_9, v_{12}\},$
$\{v_{14}, v_{15}, v_3, v_4, v_5, v_8\},$	$\{v_{10}, v_{11}, v_{15}, v_{16}, v_1, v_4\},$	$\{v_6, v_7, v_{11}, v_{12}, v_{13}, v_{16}\},$
$\{v_2, v_3, v_4, v_9, v_{12}, v_{16}\},$	$\{v_{14}, v_{15}, v_{16}, v_5, v_8, v_{12}\},$	$\{v_{10}, v_{11}, v_{12}, v_1, v_4, v_8\},$
$\{v_6, v_7, v_8, v_{13}, v_{16}, v_4\},$	$\{v_1, v_2, v_3, v_7, v_{11}, v_{13}\},$	$\{v_{13}, v_{14}, v_{15}, v_3, v_7, v_9\},$
$\{v_9, v_{10}, v_{11}, v_{15}, v_3, v_5\},$	$\{v_5, v_6, v_7, v_{11}, v_{15}, v_1\},$	$\{v_1, v_2, v_6, v_7, v_{11}, v_{13}\},$
$\{v_{13}, v_{14}, v_2, v_3, v_7, v_9\},$	$\{v_9, v_{10}, v_{14}, v_{15}, v_3, v_5\},$	$\{v_5, v_6, v_{10}, v_{11}, v_{15}, v_1\},$
$\{v_1, v_7, v_{10}, v_{11}, v_{13}, v_{14}\},$	$\{v_{13}, v_3, v_6, v_7, v_9, v_{10}\},$	$\{v_9, v_{15}, v_2, v_3, v_5, v_6\},$
$\{v_5, v_{11}, v_{14}, v_{15}, v_1, v_2\},$	$\{v_1, v_7, v_{11}, v_{13}, v_{14}, v_{15}\},$	$\{v_{13}, v_3, v_7, v_9, v_{10}, v_{11}\},$
$\{v_9, v_{15}, v_3, v_5, v_6, v_7\},$	$\{v_5, v_{11}, v_{15}, v_1, v_2, v_3\},$	$\{v_1, v_3, v_4, v_7, v_{11}, v_{13}\},$
$\{v_{13}, v_{15}, v_{16}, v_3, v_7, v_9\},$	$\{v_9, v_{11}, v_{12}, v_{15}, v_3, v_5\},$	$\{v_5, v_7, v_8, v_{11}, v_{15}, v_1\},$
$\{v_1, v_7, v_{11}, v_{13}, v_{15}, v_{16}\},$	$\{v_{13}, v_3, v_7, v_9, v_{11}, v_{12}\},$	$\{v_9, v_{15}, v_3, v_5, v_7, v_8\},$
$\{v_5, v_{11}, v_{15}, v_1, v_3, v_4\},$	$\{v_1, v_4, v_7, v_8, v_{11}, v_{13}\},$	$\{v_{13}, v_{16}, v_3, v_4, v_7, v_9\},$

$\{v_9, v_{12}, v_{15}, v_{16}, v_3, v_5\}, \{v_5, v_8, v_{11}, v_{12}, v_{15}, v_1\}, \{v_1, v_7, v_{11}, v_{12}, v_{13}, v_{16}\},$
 $\{v_{13}, v_3, v_7, v_8, v_9, v_{12}\}, \{v_9, v_{15}, v_3, v_4, v_5, v_8\}, \{v_5, v_{11}, v_{15}, v_{16}, v_1, v_4\},$
 $\{v_1, v_5, v_6, v_7, v_{11}, v_{13}\}, \{v_{13}, v_1, v_2, v_3, v_7, v_9\}, \{v_9, v_{13}, v_{14}, v_{15}, v_3, v_5\},$
 $\{v_5, v_9, v_{10}, v_{11}, v_{15}, v_1\}, \{v_1, v_5, v_7, v_8, v_{11}, v_{13}\}, \{v_{13}, v_1, v_3, v_4, v_7, v_9\},$
 $\{v_9, v_{13}, v_{15}, v_{16}, v_3, v_5\}, \{v_5, v_9, v_{11}, v_{12}, v_{15}, v_1\}, \{v_1, v_7, v_9, v_{10}, v_{11}, v_{13}\},$
 $\{v_{13}, v_3, v_5, v_6, v_7, v_9\}, \{v_9, v_{15}, v_1, v_2, v_3, v_5\}, \{v_5, v_{11}, v_{13}, v_{14}, v_{15}, v_1\},$
 $\{v_1, v_7, v_9, v_{11}, v_{12}, v_{13}\}, \{v_{13}, v_3, v_5, v_7, v_8, v_9\}, \{v_9, v_{15}, v_1, v_3, v_4, v_5\},$
 $\{v_5, v_{11}, v_{13}, v_{15}, v_{16}, v_1\}, \{v_1, v_2, v_4, v_8, v_{12}, v_{14}\}, \{v_{13}, v_{14}, v_{16}, v_4, v_8, v_{10}\},$
 $\{v_9, v_{10}, v_{12}, v_{16}, v_4, v_6\}, \{v_5, v_6, v_8, v_{12}, v_{16}, v_2\}, \{v_1, v_2, v_5, v_8, v_{12}, v_{14}\},$
 $\{v_{13}, v_{14}, v_1, v_4, v_8, v_{10}\}, \{v_9, v_{10}, v_{13}, v_{16}, v_4, v_6\}, \{v_5, v_6, v_9, v_{12}, v_{16}, v_2\},$
 $\{v_2, v_8, v_9, v_{12}, v_{13}, v_{14}\}, \{v_{14}, v_4, v_5, v_8, v_9, v_{10}\}, \{v_{10}, v_{16}, v_1, v_4, v_5, v_6\},$
 $\{v_6, v_{12}, v_{13}, v_{16}, v_1, v_2\}, \{v_2, v_8, v_{12}, v_{13}, v_{14}, v_{16}\}, \{v_{14}, v_4, v_8, v_9, v_{10}, v_{12}\},$
 $\{v_{10}, v_{16}, v_4, v_5, v_6, v_8\}, \{v_6, v_{12}, v_{16}, v_1, v_2, v_4\}, \{v_2, v_8, v_9, v_{10}, v_{12}, v_{14}\},$
 $\{v_{14}, v_4, v_5, v_6, v_8, v_{10}\}, \{v_{10}, v_{16}, v_1, v_2, v_4, v_6\}, \{v_6, v_{12}, v_{13}, v_{14}, v_{16}, v_2\},$
 $\{v_2, v_5, v_6, v_8, v_{12}, v_{14}\}, \{v_{14}, v_1, v_2, v_4, v_8, v_{10}\}, \{v_{10}, v_{13}, v_{14}, v_{16}, v_4, v_6\},$
 $\{v_6, v_9, v_{10}, v_{12}, v_{16}, v_2\}, \{v_2, v_3, v_7, v_8, v_{12}, v_{14}\}, \{v_{14}, v_{15}, v_3, v_4, v_8, v_{10}\},$
 $\{v_{10}, v_{11}, v_{15}, v_{16}, v_4, v_6\}, \{v_6, v_7, v_{11}, v_{12}, v_{16}, v_2\}, \{v_2, v_3, v_4, v_8, v_{12}, v_{14}\},$
 $\{v_{14}, v_{15}, v_{16}, v_4, v_8, v_{10}\}, \{v_{10}, v_{11}, v_{12}, v_{16}, v_4, v_6\}, \{v_6, v_7, v_8, v_{12}, v_{16}, v_2\},$
 $\{v_2, v_6, v_7, v_8, v_{12}, v_{14}\}, \{v_{14}, v_2, v_3, v_4, v_8, v_{10}\}, \{v_{10}, v_{14}, v_{15}, v_{16}, v_4, v_6\},$
 $\{v_6, v_{10}, v_{11}, v_{12}, v_{16}, v_2\}, \{v_2, v_8, v_{11}, v_{12}, v_{14}, v_{15}\}, \{v_{14}, v_4, v_7, v_8, v_{10}, v_{11}\},$
 $\{v_{10}, v_{16}, v_3, v_4, v_6, v_7\}, \{v_6, v_{12}, v_{15}, v_{16}, v_2, v_3\}, \{v_2, v_8, v_{12}, v_{14}, v_{15}, v_{16}\},$
 $\{v_{14}, v_4, v_8, v_{10}, v_{11}, v_{12}\}, \{v_{10}, v_{16}, v_4, v_6, v_7, v_8\}, \{v_6, v_{12}, v_{16}, v_2, v_3, v_4\},$
 $\{v_2, v_8, v_{10}, v_{11}, v_{12}, v_{14}\}, \{v_{14}, v_4, v_6, v_7, v_8, v_{10}\}, \{v_{10}, v_{16}, v_2, v_3, v_4, v_6\},$
 $\{v_6, v_{12}, v_{14}, v_{15}, v_{16}, v_2\}.$

Two vertices u and v are said to be first associated if $d(u, v) = 1$, second associated if $d(u, v) = 2$, third associated if $d(u, v) = 3$ and fourth associated if $d(u, v) = 4$. The first, second, third and fourth associates of each vertices are shown in the Table 6.

With this association, the parameters of second kind are given by $n_1 = 4, n_2 = 6, n_3 = 4$ and $n_4 = 1$ with

$$\begin{aligned}
 P^1 &= \begin{pmatrix} p_{11}^1 & p_{12}^1 & p_{13}^1 & p_{14}^1 \\ p_{21}^1 & p_{22}^1 & p_{23}^1 & p_{24}^1 \\ p_{31}^1 & p_{32}^1 & p_{33}^1 & p_{34}^1 \\ p_{41}^1 & p_{42}^1 & p_{43}^1 & p_{44}^1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
 P^2 &= \begin{pmatrix} p_{11}^2 & p_{12}^2 & p_{13}^2 & p_{14}^2 \\ p_{21}^2 & p_{22}^2 & p_{23}^2 & p_{24}^2 \\ p_{31}^2 & p_{32}^2 & p_{33}^2 & p_{34}^2 \\ p_{41}^2 & p_{42}^2 & p_{43}^2 & p_{44}^2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 4 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
 P^3 &= \begin{pmatrix} p_{11}^3 & p_{12}^3 & p_{13}^3 & p_{14}^3 \\ p_{21}^3 & p_{22}^3 & p_{23}^3 & p_{24}^3 \\ p_{31}^3 & p_{32}^3 & p_{33}^3 & p_{34}^3 \\ p_{41}^3 & p_{42}^3 & p_{43}^3 & p_{44}^3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \\
 P^4 &= \begin{pmatrix} p_{11}^4 & p_{12}^4 & p_{13}^4 & p_{14}^4 \\ p_{21}^4 & p_{22}^4 & p_{23}^4 & p_{24}^4 \\ p_{31}^4 & p_{32}^4 & p_{33}^4 & p_{34}^4 \\ p_{41}^4 & p_{42}^4 & p_{43}^4 & p_{44}^4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 & 0 \\ 0 & 6 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
 \end{aligned}$$

Table 6 1st, 2nd, 3rd and 4th associates of hypercube Q_4

Elements	First associates	Second associates	Third associates	Fourth associates
v_1	v_2, v_4, v_5, v_{13}	$v_3, v_6, v_8, v_9, v_{14}, v_{16}$	$v_7, v_{10}, v_{12}, v_{15}$	v_{11}
v_2	v_1, v_3, v_6, v_{14}	$v_4, v_5, v_7, v_{10}, v_{13}, v_{15}$	v_8, v_9, v_{11}, v_{16}	v_{12}
v_3	v_2, v_4, v_7, v_{15}	$v_1, v_6, v_8, v_{11}, v_{14}, v_{16}$	$v_5, v_{10}, v_{12}, v_{13}$	v_9
v_4	v_1, v_3, v_8, v_{16}	$v_2, v_5, v_7, v_{12}, v_{13}, v_{15}$	v_6, v_9, v_{11}, v_{14}	v_{10}
v_5	v_1, v_6, v_8, v_9	$v_2, v_4, v_7, v_{10}, v_{12}, v_{13}$	$v_3, v_{11}, v_{14}, v_{16}$	v_{15}
v_6	v_2, v_5, v_7, v_{10}	$v_1, v_3, v_8, v_9, v_{11}, v_{14}$	$v_4, v_{12}, v_{13}, v_{15}$	v_{16}
v_7	v_3, v_6, v_8, v_{11}	$v_2, v_4, v_5, v_{10}, v_{12}, v_{15}$	v_1, v_9, v_{14}, v_{16}	v_{13}
v_8	v_4, v_5, v_7, v_{12}	$v_1, v_3, v_6, v_9, v_{11}, v_{16}$	$v_2, v_{10}, v_{13}, v_{15}$	v_{14}
v_9	$v_5, v_{10}, v_{12}, v_{13}$	$v_1, v_6, v_8, v_{11}, v_{14}, v_{16}$	v_2, v_4, v_7, v_{15}	v_3
v_{10}	v_6, v_9, v_{11}, v_{14}	$v_2, v_5, v_7, v_{12}, v_{13}, v_{15}$	v_1, v_3, v_8, v_{16}	v_4
v_{11}	$v_7, v_{10}, v_{12}, v_{15}$	$v_3, v_6, v_8, v_9, v_{14}, v_{16}$	v_2, v_4, v_5, v_{13}	v_1
v_{12}	v_8, v_9, v_{11}, v_{16}	$v_4, v_5, v_7, v_{10}, v_{13}, v_{15}$	v_1, v_3, v_6, v_{14}	v_2
v_{13}	v_1, v_9, v_{14}, v_{16}	$v_2, v_4, v_5, v_{10}, v_{12}, v_{15}$	v_3, v_6, v_8, v_{11}	v_7
v_{14}	$v_2, v_{10}, v_{13}, v_{15}$	$v_1, v_3, v_6, v_9, v_{11}, v_{16}$	v_4, v_5, v_7, v_{12}	v_8
v_{15}	$v_3, v_{11}, v_{14}, v_{16}$	$v_2, v_4, v_7, v_{10}, v_{12}, v_{13}$	v_1, v_6, v_8, v_9	v_5
v_{16}	$v_4, v_{12}, v_{13}, v_{15}$	$v_1, v_3, v_8, v_9, v_{11}, v_{14}$	v_2, v_5, v_7, v_{10}	v_6

Thus the parameters of first kind are given by $v = 16, b = 196, r = 72, s = 6, \lambda_1 = 30, \lambda_2 = 16, \lambda_3 = 24$ and $\lambda_4 = 48$. $\square$

In this section, we pose the following Open problem.

Open Problem: Find the collection of all γ_{bc} -sets of a hypercube Q_n with $n \geq 5$ and form PBIB-designs with $m \geq 5$ -association schemes (if possible generalize).

7 Cycles with $m = \lfloor \frac{p}{2} \rfloor$ association schemes

A cycle C_p is a closed path of simple graph with $p \geq 3$ vertices.

To prove our next result, we consider $x \pmod p = x$ if $x \leq p$, and $x \pmod p = p - 2$, if $x > p$, where $p - 2$ is the remainder when x is divided by p .

Theorem 7.1 *The collection of all γ_{bc} -sets of C_p , $p \geq 3$ forms a PBIB-design with $\lfloor \frac{p}{2} \rfloor$ association scheme and parameters $v = p, b = p, r = p - 2, s = p - 2, \lambda_1 = p - 3$ and $\lambda_i = p - 4$ for $2 \leq i \leq \lfloor \frac{p}{2} \rfloor$.*

Proof Let a cycle C_p be labeled as $v_1, v_2, \dots, v_p$ with the collection of all γ_{bc} -sets are given by $V(C_p) - \{v_i, v_{(i+1) \pmod p}\}, 1 \leq i \leq p$. Then two vertices u, v are said to be k th ($1 \leq k \leq \lfloor \frac{p}{2} \rfloor$) associated if $d(u, v) = k$. Thus the following cases arise:

Case 1. If p is even, then the $1 \leq i, j, k \leq \frac{p}{2}$ associates of each elements are given in Table 7.

Table 7 $\frac{p}{2}$ th associates of even cycle

Elements	First associates	Second associates	k th associates	$\frac{p}{2}$ th associates
v_1	v_p, v_2	v_{p-1}, v_3	$v_{(p-(k-1))(\bmod p)}, v_{(1+k)(\bmod p)}$	$v_{1+\frac{p}{2}}$
v_2	v_1, v_3	v_p, v_4	$v_{(p-(k-2))(\bmod p)}, v_{(2+k)(\bmod p)}$	$v_{2+\frac{p}{2}}$
v_3	v_2, v_4	v_1, v_5	$v_{(p-(k-3))(\bmod p)}, v_{(3+k)(\bmod p)}$	$v_{3+\frac{p}{2}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_i	$v_{(i-1)(\bmod p)},$ $v_{(i+1)(\bmod p)}$	$v_{(i-2)(\bmod p)},$ $v_{(i+2)(\bmod p)}$	$v_{(p-(k-i))(\bmod p)}, v_{(i+k)(\bmod p)}$	$v_{(i+\frac{p}{2})(\bmod p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	v_{p-1}, v_1	v_{p-2}, v_2	v_{p-k}, v_k	$v_{\frac{p}{2}}$

With this association scheme, the parameters of second kind are given by $n_i = 2$ for $1 \leq i \leq \frac{p}{2} - 1$, $n_{\frac{p}{2}} = 1$ and

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \cdots & p_{1\frac{p}{2}}^k \\ p_{21}^k & p_{22}^k & \cdots & p_{2\frac{p}{2}}^k \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ p_{\frac{p}{2}1}^k & p_{\frac{p}{2}2}^k & \cdots & p_{\frac{p}{2}\frac{p}{2}}^k \end{pmatrix}$$

with entries given as follows:

Now consider all possible values of k .

If $k = 1$, then

$$p_{ij}^1 = 1 \text{ for } 1 \leq i \leq \frac{p}{2} - 1, j = i + 1;$$

$$p_{ij}^1 = 1 \text{ for } i = 1 + j, 1 \leq j \leq \frac{p}{2} - 1.$$

If $2 \leq k \leq \frac{p}{2} - 1$, then

$$p_{ij}^k = 1 \text{ for } 1 \leq i \leq \frac{p}{2} - 1, i + j = k, j = k + i \text{ and } i + j = p - k;$$

$$p_{ij}^k = 1 \text{ for } 1 \leq j \leq \frac{p}{2} - 1, i = k + j \text{ and } i + j = p - k.$$

If $k = \frac{p}{2}$, then

$$p_{ij}^k = 2 \text{ for } 1 \leq i \leq \frac{p}{2} - 1, j = k - i \text{ and the remaining entries are zero.}$$

Hence the parameters of first kind are given by $v = p$, $b = p$, $s = p - 2$, $r = p - 2$, $\lambda_1 = p - 3$ and $\lambda_i = p - 4$ for $2 \leq i \leq \frac{p}{2}$.

Case 2. If p is odd, then the $1 \leq i, j, k \leq \left(\frac{p-1}{2}\right)$ th associates of each vertices are given in the Table 8.

Table 8 $\left(\frac{p-1}{2}\right)$ th associates of odd cycles

Elements	First associates	Second associates	k th associates	$\left(\frac{p-1}{2}\right)$ th associates
v_1	v_p, v_2	v_{p-1}, v_3	$v_{p-(k-1)}, v_{1+k}$	$v_{1+\frac{p-1}{2}}, v_{1+\frac{p-1}{2}+1}$
v_2	v_1, v_3	v_p, v_4	$v_{p-(k-2)}, v_{2+k}$	$v_{2+\frac{p-1}{2}}, v_{2+\frac{p-1}{2}+1}$
v_3	v_2, v_4	v_1, v_5	$v_{p-(k-3)}, v_{3+k}$	$v_{3+\frac{p-1}{2}}, v_{3+\frac{p-1}{2}+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_i	$v_{(i-1)(\bmod p)},$ $v_{(i+1)(\bmod p)}$	$v_{(i-2)(\bmod p)},$ $v_{(i+2)(\bmod p)}$	$v_{(p-(k-i))(\bmod p)},$ $v_{(i+k)(\bmod p)}$	$v_{(i+\frac{p-1}{2})(\bmod p)},$ $v_{(i+\frac{p-1}{2}+1)(\bmod p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	v_{p-1}, v_1	v_{p-2}, v_2	v_{p-k}, v_k	$v_{\frac{p-1}{2}}, v_{\frac{p-1}{2}+1}$

With this association scheme, the parameters of second kind are given by $n_i = 2$ for $1 \leq i \leq \frac{p-1}{2}$ and

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \cdots & p_{1\frac{p-1}{2}}^k \\ p_{21}^k & p_{22}^k & \cdots & p_{2\frac{p-1}{2}}^k \\ \vdots & \vdots & & \vdots \\ p_{\left(\frac{p-1}{2}\right)1}^k & p_{\left(\frac{p-1}{2}\right)2}^k & \cdots & p_{\left(\frac{p-1}{2}\right)\left(\frac{p-1}{2}\right)}^k \end{pmatrix}$$

with entries given as follows:

Now consider all possible values of k .

If $k = 1$, then

$$p_{ij}^1 = 1 \text{ for } 1 \leq i \leq \frac{p-1}{2} - 1, j = i + 1;$$

$$p_{ij}^1 = 1 \text{ for } 1 \leq j \leq \frac{p-1}{2} - 1, i = 1 + j;$$

$$p_{ij}^1 = 1 \text{ for } i = \frac{p-1}{2}, j = \frac{p-1}{2}.$$

If $2 \leq k \leq \frac{p-3}{2}$, then

$$p_{ij}^k = 1 \text{ for } 1 \leq i \leq \frac{p-3}{2}, i + j = k, j = k + i \text{ and } i + j = p - k;$$

$$p_{ij}^k = 1 \text{ for } 1 \leq j \leq \frac{p-3}{2}, i = k + j \text{ and } i + j = p - k.$$

If $k = \frac{p-1}{2}$, then

$$p_{ij}^k = 1 \text{ for } 1 \leq i \leq \frac{p-3}{2}, j = \frac{p-1}{2} - i;$$

$$p_{ij}^k = 1 \text{ for } 1 \leq i \leq \frac{p-1}{2}, j = \frac{p+1}{2} - i \text{ and the remaining entries are zero.}$$

Hence the parameters of first kind are given by $v = p$, $b = p$, $s = p - 2$, $r = p - 2$, $\lambda_1 = p - 3$ and $\lambda_i = p - 4$ for $2 \leq i \leq \frac{p-1}{2}$. $\square$

Acknowledgements We are immensely grateful to Prof. N. D. Soner for sharing his pearls of wisdom with us, that has greatly improved the manuscript. The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

References

1. Brouwer, A.E., Ivanov, A.V., Klin, M.H.: Some new strongly regular graphs. *Combinatorica* **9**, 339–344 (1989)
2. Bose, R.C., Nair, K.R.: Partially balanced incomplete block designs. *Sankhya* **4**, 337–372 (1939)
3. Cyman, J.: The outer-connected domination number of a graph. *Australas. J. Comb.* **28**, 35–46 (2007)
4. Cyman, J., Lemanska, M., Raczek, Joanna: On the doubly connected domination number of a graph. *Cent. Eur. J. Math.* **4**(1), 34–35 (2006)
5. Colbourn, C.J., Dinitz, J.H. (eds.): *Handbook of Combinatorial Designs*. CRC Press, Boca Raton (2007)
6. Harary, F.: *Graph Theory*. Addison-Wesley, Reading (1969)
7. Haynes, T.W., Hedetniemi, S.T., Slater, P.J.: *Fundamentals of Domination in Graphs*. Marcel Dekker Inc, New York (1998)
8. Hedetniemi, S.T., Laskar, R.: Connected domination in graphs. In: Bollobás, B. (ed.) *Graph Theory and Combinatorics*, pp. 209–217. Academic press, London (1984)
9. Kulli, V.R., Janakiram, B.: The nonsplit domination number of a graph. *Indian J. Pure Appl. Math.* **31**(4), 441–447 (2000)
10. Manjunath, N., Deepak, G., Chaluvvaraju, B.: PBIB-Designs and Association schemes arising from minimum connected dominating sets of a graph. *South Asian J. Math.* **4**(5), 225–232 (2014)
11. Petersen, J.: Die theorie der regularen graphen. *Acta Math.* **15**(1), 193–220 (1891)
12. Sampathkumar, E., Walikar, H.B.: The connected domination number of a graph. *J. Math. Phys. Sci.* **13**, 607–613 (1979)
13. Vartak, M.N.: The non-existence of certain PBIB designs. *Ann. Math. Statist.* **30**, 1051–1062 (1959)
14. Walikar, H.B., Ramane, H.S., Acharya, B.D., Shekarappa, H.S., Arumugam, S.: Partially balanced incomplete block design arising from minimum dominating sets of paths and cycles. *AKCE J. Graphs Comb.* **4**(2), 223–232 (2007)