Some New Congruences for Andrews' Singular Overpartition Pairs

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Abstract In a recent work, Andrews defined the combinatorial objects called singular overpartitions denoted by $\overline{C}_{k,i}(n)$, which count the number of overpartitions of n in which no part is divisible by k and only parts congruent to $\pm i$ modulo k may be overlined. Many authors have found congruences and infinite families of congruences modulo powers of 2 and 3. In this paper, we find some new infinite families of congruences for $\overline{C}_{1,2}^6(n)$ modulo 27 and congruences modulo 4 for $\overline{C}_{1,5}^{12}(n)$, $\overline{C}_{3,3}^9(n)$ and $\overline{C}_{5,5}^{15}(n)$.

Keywords Congruences · Dissections · Singular overpartition pairs · Theta function

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1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n . An overpartition, introduced by Corteel and Lovejoy [9], of a non-negative integer n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. For example, the eight overpartitions of 3 are

$$3, \quad \overline{3}, \quad 2 + 1, \quad \overline{2} + 1, \quad 2 + \overline{1}, \quad \overline{2} + \overline{1}, \quad 1 + 1 + 1, \quad \overline{1} + 1 + 1.$$

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Andrews [3] introduced singular overpartitions. To introduce singular overpartitions, first he defined some properties of the entries in a Frobenius symbol for n , which is of the form

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}, \quad (1)$$

where $\sum(a_i + b_i + 1) = n$ and $a_1 > a_2 > \dots > a_r \geq 0, b_1 > b_2 > \dots > b_r \geq 0$. There is a natural mapping that reveals a one-to-one correspondence between the Frobenius symbols for n and the ordinary partitions of n . “Singular overpartitions” are Frobenius symbols for n with at most one overlined entry in each row. More precisely, for two positive integers k and i , a column $\begin{pmatrix} a_j \\ b_j \end{pmatrix}$ in a Frobenius symbol is (k, i) -positive if $a_j - b_j \geq k - i - 1$ and (k, i) -negative if $a_j - b_j \leq -i + 1$. If $-i + 1 < a_j - b_j < k - i + 1$, then we say the column is (k, i) -neutral. Two columns have the same parity if they are both (k, i) -positive or (k, i) -negative. We can divide the Frobenius symbol into (k, i) -blocks such that all the entries in each block have either the same (k, i) -parity or (k, i) -neutral. The first non-neutral column in each parity block is called the anchor of the block. A (k, i) -parity block is neutral if all columns in it are neutral and a (k, i) -parity block is positive (resp. negative) if it contains no (k, i) -negative (resp. positive) columns.

A Frobenius symbol is (k, i) -singular if

- 1) there are no overlined entries, or
- 2) the one overlined entry on the top row occurs in an anchor of a (k, i) -positive block, or
- 3) the one overlined entry on the bottom row occurs in an anchor of a (k, i) -negative block, and
- 4) if there is one overlined entry in each row, then they occur in adjacent (k, i) -parity blocks.

Let $\overline{Q}_{k,i}(n)$ denote the number of such singular overpartitions for $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$. Andrews proved that $\overline{Q}_{k,i}(n) = \overline{C}_{k,i}(n)$, where $\overline{C}_{k,i}(n)$ counts the number of overpartitions of n in which no part is divisible by k and only parts congruent to $\pm i$ modulo k may be overlined. Therefore for $k \geq 3$ and $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{Q}_{k,i}(n) q^n = \sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n = \frac{(q^k; q^k)_{\infty} (-q^i; q^k)_{\infty} (-q^{k-i}; q^k)_{\infty}}{(q; q)_{\infty}}. \quad (2)$$

The ten singular overpartitions counted by $\overline{C}_{3,1}(4)$ are

$$4, \overline{4}, 2 + 2, \overline{2} + 2, 2 + 1 + 1, \overline{2} + 1 + 1, 2 + \overline{1} + 1, \overline{2} + \overline{1} + 1, 1 + 1 + 1 + 1, \overline{1} + 1 + 1 + 1.$$

Throughout the paper, we use the standard q -series notation, and f_k is defined as

$$f_k := (q^k; q^k)_{\infty} = \lim_{n \rightarrow \infty} \prod_{m=1}^n (1 - q^{mk}).$$

For $|ab| < 1$, Ramanujan’s general theta function $f(a, b)$ is defined as

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

where the product representations arise from Jacobi’s triple product identity [5, p. 35, Entry 19], [1]

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}.$$

The special cases of $f(a, b)$ are

$$\begin{aligned}\varphi(q) &:= f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} = \frac{f_2^5}{f_1^2 f_4^2}, \\ \psi(q) &:= f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \frac{f_2^2}{f_1},\end{aligned}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty} = f_1, \quad (3)$$

Andrews [3] proves that, for all $n \geq 0$, $\overline{C}_{3,1}(n) = \overline{A}_3(n)$, where $\overline{A}_3(n)$ is the number of overpartitions of n into parts not divisible by 3. The function $\overline{A}_{\ell}(n)$, which counts the number of overpartitions of n into parts not divisible by ℓ , plays a key role in the work of Lovejoy [13].

In his beautiful paper, Andrews also found the following congruences:

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}. \quad (4)$$

Chen et al. [8] have generalized (4) and proved some congruences modulo 2, 3, 4, and 8 for $\overline{C}_{3,1}(n)$. They also proved some congruence for $\overline{C}_{4,1}(n)$, $\overline{C}_{6,1}(n)$ and $\overline{C}_{6,2}(n)$ modulo powers of 2 and 3. More recently Ahmed and Baruah [2] have found some new congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{8,2}(n)$, $\overline{C}_{12,4}(n)$, $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$ modulo 18, 36. Chen [7] has also found some congruences modulo powers of 2 for $\overline{C}_{3,1}(n)$, $\overline{C}_{4,1}(n)$. Yao [22] has proved congruences modulo 16, 32, 64 for $\overline{C}_{3,1}(n)$. Naika and Gireesh [14] have found some congruences modulo 6, 12, 16, 18, 24, 48, and 72 for $\overline{C}_{3,1}(n)$. Naika and Nayaka [16] have proved some congruences for $\overline{CO}_{3,1}(n)$ modulo powers of 2 and 3. They have also proved in the paper [17] modulo 4 for $\overline{C}_{4,1}^3(n)$ and $\overline{C}_{4,1}^5(n)$.

Mahadeva Naika and Shivashankar [18] have defined the Andrews' singular overpartition pairs of n . Let $\overline{C}_{i,j}^{\delta}(n)$ denote the number of Andrews' singular overpartition pairs of n in which no part is divisible by δ and only parts congruent to $\pm i, \pm j$ modulo δ may be overlined. An Andrews' singular overpartition pair π of n is a pair of Andrews' singular overpartitions (λ, μ) such that the sum of all of the parts is n . For $\delta \geq 3$ and $1 \leq i, j \leq \lfloor \frac{\delta}{2} \rfloor$, the generating function for $\overline{C}_{i,j}^{\delta}(n)$ is

$$\sum_{n=0}^{\infty} \overline{C}_{i,j}^{\delta}(n) q^n = \frac{f(q^i, q^{\delta-i}) f(q^j, q^{\delta-j})}{(q; q)_{\infty}^2}. \quad (5)$$

In this paper, we find some new infinite families of congruences for $\overline{C}_{1,2}^6(n)$ modulo 27 and congruences modulo 4 for $\overline{C}_{1,5}^{12}(n)$, $\overline{C}_{3,3}^9(n)$ and $\overline{C}_{5,5}^{15}(n)$. The main results of this paper can be stated as follows.

Theorem 1 For any $\alpha \geq 0$ and $n \geq 0$,

$$\overline{C}_{1,2}^6 \left(4^{\alpha+2}n + \frac{4^{\alpha+2} - 1}{3} \right) \equiv 7^{\alpha+1} \cdot \overline{C}_{1,2}^6(4n+1) \pmod{27}. \quad (6)$$

Theorem 2 For all $\alpha \geq 0$ and $n \geq 0$,

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(3n)q^n \equiv \frac{f_1 f_3 f_6}{f_2} \pmod{4}, \quad (7)$$

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(3n+1)q^n \equiv 3 \frac{f_1^2 f_6^4}{f_2^2 f_3^2} \pmod{4}, \quad (8)$$

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(3n+2)q^n \equiv 3 \frac{f_1 f_3 f_{12}^3}{f_4 f_6^2} \pmod{4}, \quad (9)$$

$$\bar{C}_{1,5}^{12}(3^{\alpha+1}n + 3^{\alpha+1} - 1) \equiv 3^{\alpha+1} \cdot \bar{C}_{1,5}^{12}(n) \pmod{4}. \quad (10)$$

Theorem 3 For all integers $\alpha \geq 0$ and $n \geq 0$,

$$\bar{C}_{1,5}^{12}(12n+6) \equiv 0 \pmod{4}, \quad (11)$$

$$\bar{C}_{1,5}^{12}(48n+27) \equiv 0 \pmod{4}, \quad (12)$$

$$\bar{C}_{1,5}^{12}(96n+87) \equiv 0 \pmod{4}, \quad (13)$$

$$\bar{C}_{1,5}^{12}(3 \cdot 4^{\alpha+3}n + 7 \cdot 4^{\alpha+2} - 1) \equiv 0 \pmod{4}. \quad (14)$$

Theorem 4 For all $\alpha \geq 0$ and $n \geq 0$,

$$\bar{C}_{1,5}^{12}(24n+13) \equiv 0 \pmod{4}, \quad (15)$$

$$\bar{C}_{1,5}^{12}(24n+3) \equiv \bar{C}_{1,5}^{12}(12n+1) \pmod{4}, \quad (16)$$

$$\bar{C}_{1,5}^{12}(24n+15) \equiv \bar{C}_{1,5}^{12}(12n+7) \pmod{4}. \quad (17)$$

Theorem 5 For each $n \geq 0$ and $\alpha \geq 0$,

$$\bar{C}_{1,5}^{12}(12 \cdot 25^{\alpha+1}n + 25^{\alpha+1} - 1) \equiv \bar{C}_{1,5}^{12}(12n) \pmod{4}, \quad (18)$$

$$\bar{C}_{1,5}^{12}(60(5n+i) + 24) \equiv 0 \pmod{4}, \quad (19)$$

where $i = 1, 2, 3, 4$.

Theorem 6 Let p be a prime ≥ 5 , $\left(\frac{-4}{p}\right) = -1$. Then for all integers $\alpha \geq 1$, and $n \geq 0$,

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(192p^{2\alpha}n + 39p^{2\alpha} - 1)q^n \equiv 2f_1 f_4 \pmod{4}. \quad (20)$$

Theorem 7 Let p be a prime ≥ 5 , $\left(\frac{-4}{p}\right) = -1$. Then for all integers $\alpha \geq 0$, and $n \geq 0$,

$$\bar{C}_{1,5}^{12}(192p^{2\alpha+2}n + 192p^{2\alpha+1}i + 40p^{2\alpha+2} - 1) \equiv 0 \pmod{4},$$

where i is an integer and $1 \leq i \leq p-1$.

Theorem 8 Let p be a prime ≥ 5 , $\left(\frac{-6}{p}\right) = -1$. Then for all integers $\alpha \geq 1$, and $n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^9 \left(4p^{2\alpha}n + \frac{7p^{2\alpha} - 1}{6} \right) q^n \equiv 2f_1f_6 \pmod{4}. \quad (21)$$

Theorem 9 Let p be a prime ≥ 5 , $\left(\frac{-6}{p}\right) = -1$. Then for all integers $\alpha \geq 0$, and $n \geq 0$,

$$\overline{C}_{3,3}^9 \left(4p^{2\alpha+2}n + 4p^{2\alpha+1}i + \frac{7p^{2\alpha+2} - 1}{6} \right) \equiv 0 \pmod{4}, \quad (22)$$

where i is an integer and $1 \leq i \leq p-1$.

Theorem 10 For each $n \geq 0$ and $\alpha \geq 0$,

$$\overline{C}_{5,5}^{15}(16n+9) \equiv 0 \pmod{4}, \quad (23)$$

$$\overline{C}_{5,5}^{15}(20n+3) \equiv \overline{C}_{5,5}^{15}(8n+1) \pmod{4}, \quad (24)$$

$$\overline{C}_{5,5}^{15}(10n+3) \equiv \overline{C}_{5,5}^{15}(4n+1) \pmod{4}, \quad (25)$$

$$\overline{C}_{5,5}^{15} \left(2 \cdot 4^{\alpha+1}n + \frac{4^{\alpha+2} - 1}{3} \right) \equiv \overline{C}_{5,5}^{15}(2n+1) \pmod{4}, \quad (26)$$

$$\overline{C}_{5,5}^{15} \left(4 \cdot 5^{5\alpha+6}(5n+i) + \frac{2 \cdot 5^{5\alpha+6} - 1}{3} \right) \equiv 0 \pmod{4}, \quad (27)$$

where $i = 1, 2, 3, 4$.

$$\overline{C}_{5,5}^{15} \left(2 \cdot 5^{2\alpha+4}(5n+j) + \frac{4 \cdot 5^{2\alpha+4} - 1}{3} \right) \equiv 0 \pmod{4}, \quad (28)$$

where $j = 3, 4$.

2 Preliminary Results

We need the following few dissection formulas to prove our main results,

Lemma 1 [19, p. 212] We have the following 5-dissection

$$f_1 = f_{25} \left(a(q^5) - q - \frac{q^2}{a(q^5)} \right), \quad (29)$$

where

$$a(q) := \frac{(q^2, q^3; q^5)_{\infty}}{(q, q^4; q^5)_{\infty}}.$$

Lemma 2 *The following 2-dissections hold:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}, \quad (30)$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \quad (31)$$

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q \frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \quad (32)$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q \frac{f_2^2 f_{12}^3}{f_4 f_6^2}. \quad (33)$$

Hirschhorn et al. [11] have proved (30). For the proof of (31), see [4]. The proofs of (32) and (33) follow by changing q to $-q$ in (30) and (31) respectively with $(-q; -q)_\infty = \frac{f_2^3}{f_1 f_4}$.

Lemma 3 *The following 2-dissection formulas hold:*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q \frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \quad (34)$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q \frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \quad (35)$$

Equation (34) was proved by Baruah and Ojah [6]. Replacing q by $-q$ in (34) and using the fact that $(-q; -q)_\infty = \frac{f_2^3}{f_1 f_4}$, we get (35).

Lemma 4 *The following 2-dissection formulas hold:*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \quad (36)$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q \frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}. \quad (37)$$

Xia and Yao [21] proved (36) by employing an addition formula for theta functions. Replacing q by $-q$ in (36) and using the fact that $(-q; -q)_\infty = \frac{f_2^3}{f_1 f_4}$, we get (37).

Lemma 5 *The following 3-dissection holds:*

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q \frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^3}. \quad (38)$$

Equation (38) is nothing but Lemma 2.6 in [6].

Lemma 6 [20] *We have the following 3-dissection*

$$\frac{f_1}{f_4} = \frac{f_6 f_9 f_{18}}{f_{12}^3} - q \frac{f_3 f_{18}^4}{f_{12}^3 f_9^2} - q^2 \frac{f_6^2 f_9 f_{36}^3}{f_{12}^4 f_{18}^2}. \quad (39)$$

Proof From (3), we have

$$\begin{aligned} \frac{f_1}{f_4} &= \frac{(q; q)_\infty}{(q^4; q^4)_\infty} = \frac{(q, q^2, q^3, q^4, q^5, q^6; q^6)_\infty}{(q^4; q^4)_\infty} \\ &= \frac{(q, q^5, q^6; q^6)_\infty (q^2, q^4, q^8, q^{10}; q^{12})_\infty (q^3; q^6)_\infty (q^6; q^6)_\infty}{(q^4, q^8, q^{12}; q^{12})_\infty (q^6; q^6)_\infty} \\ &= \frac{(q, q^5, q^6; q^6)_\infty (q^2, q^{10}, q^{12}; q^{12})_\infty (q^3; q^3)_\infty}{(q^6; q^6)_\infty (q^{12}; q^{12})_\infty^2}, \end{aligned} \quad (40)$$

where $(a_1, a_2, \dots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \dots (a_k; q)_\infty$.

Now by Jacobi triple product identity

$$\begin{aligned} &(q, q^5, q^6; q^6)_\infty (q^2, q^{10}, q^{12}; q^{12})_\infty \\ &= \sum_{m=-\infty}^{\infty} (-1)^m q^{3m^2+2m} \cdot \sum_{n=-\infty}^{\infty} (-1)^n q^{6n^2+4n} \\ &= \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{3m^2+2m+6n^2+4n} \\ &= \sum_{t,u=-\infty}^{\infty} (-1)^{2t-u} q^{3(t-2u)^2+2(t-2u)+6(t+u-1)^2+4(t+u-1)} \\ &\quad + \sum_{t,u=-\infty}^{\infty} (-1)^{2t-u-1} q^{3(t-2u-1)^2+2(t-2u-1)+6(t+u)^2+4(t+u)} \\ &\quad + \sum_{t,u=-\infty}^{\infty} (-1)^{2t-u-1} q^{3(t-2u)^2+2(t-2u)+6(t+u-1)^2+4(t+u-1)} \\ &= \sum_{t,u=-\infty}^{\infty} (-1)^u q^{9t^2+18u^2+6t} + q \sum_{t,u=-\infty}^{\infty} (-1)^{-u-1} q^{9t^2+18u^2+12u} \\ &\quad + q^2 \sum_{t,u=-\infty}^{\infty} (-1)^{-u-1} q^{9t^2-6t+18u^2-12u} \\ &= (-q^3, -q^{15}, q^{18}; q^{18})_\infty (q^{18}, q^{18}, q^{36}; q^{36})_\infty \\ &\quad - q(-q^9, -q^9, q^{18}; q^{18})_\infty (q^6, q^{30}, q^{36}; q^{36})_\infty \\ &\quad - q^2(-q^3, -q^{15}, q^{18}; q^{18})_\infty (q^6, q^{30}, q^{36}; q^{36})_\infty \\ &= \frac{(q^6; q^6)_\infty (q^9; q^9)_\infty (q^{36}; q^{36})_\infty}{(q^3; q^3)_\infty (q^{12}; q^{12})_\infty (q^{18}; q^{18})_\infty} \cdot \frac{(q^{18}; q^{18})_\infty^2 (q^{36}; q^{36})_\infty}{(q^{36}; q^{36})_\infty^2} \\ &\quad - q \frac{(q^6; q^6)_\infty (q^{36}; q^{36})_\infty^2}{(q^{12}; q^{12})_\infty (q^{18}; q^{18})_\infty} \cdot \frac{(q^{18}; q^{18})_\infty^5}{(q^9; q^9)_\infty^2 (q^{36}; q^{36})_\infty^2} \\ &\quad - q^2 \frac{(q^6; q^6)_\infty^2 (q^9; q^9)_\infty (q^{36}; q^{36})_\infty}{(q^3; q^3)_\infty (q^{12}; q^{12})_\infty (q^{18}; q^{18})_\infty} \cdot \frac{(q^6; q^6)_\infty (q^{36}; q^{36})_\infty^2}{(q^{12}; q^{12})_\infty (q^{18}; q^{18})_\infty}. \end{aligned} \quad (41)$$

Substituting (41) into (40), we obtain (39). $\square$

Lemma 7 *The following 2-dissection holds:*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \quad (42)$$

Equation (42) was proved by Hirschhorn and Sellers [12].

Lemma 8 [15, Lemma 2.3] *The following 2-dissection formulas hold:*

$$f_1 f_5^3 = 2q^2 f_4 f_{20}^3 + f_2^3 f_{10} - 2q^3 \frac{f_4^4 f_{40}^2 f_{10}}{f_2 f_8^2} - q \frac{f_2^2 f_{10}^2 f_{20}}{f_4}, \quad (43)$$

$$f_1^3 f_5 = 2q^2 \frac{f_4^6 f_{40}^2 f_{10}}{f_2 f_8^2 f_{20}^2} + \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3. \quad (44)$$

Lemma 9 (Cui and Gu [10, Theorem 2.2]) *For any prime $p \geq 5$,*

$$f_1 = \sum_{\substack{k=\frac{1-p}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2},$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6} & \text{if } p \equiv 1 \pmod{6}, \\ -\frac{p-1}{6} & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

3 Proof of Theorem 1

Setting $i = 1$, $j = 2$ and $\delta = 6$ in (5), we see that

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(n) q^n = \frac{f(q, q^5) f(q^2, q^4)}{(q; q)_{\infty}^2}.$$

By the definition of $f(a, b)$ and the well-known Jacobi triple product identity, we get

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(n) q^n = \frac{f_2 f_3 f_6}{f_1^3}. \quad (45)$$

Substituting (31) into (45), we have

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(n) q^n = f_2 f_6 \left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7} \right).$$

Equating odd parts of the above equation, we obtain

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(2n+1) q^n = 3 \frac{f_2^2 f_3^2 f_6^2}{f_1^6}. \quad (46)$$

Employing (31) into (46), we arrive at

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(2n+1) q^n = 3 \frac{f_4^{12} f_6^8}{f_2^{16} f_{12}^4} + 18q \frac{f_4^8 f_6^6}{f_2^{14}} + 27q^2 \frac{f_4^4 f_6^4 f_{12}^4}{f_2^{12}}. \quad (47)$$

Extracting the terms involving q^{2n} from both sides of (47), we have

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(4n+1)q^n = 3 \frac{f_2^{12} f_3^8}{f_1^{16} f_6^4} + 27q \frac{f_2^4 f_3^4 f_6^4}{f_1^{12}},$$

which implies that

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(4n+1)q^n \equiv 3 \frac{f_2^{12} f_3^8}{f_1^{16} f_6^4} \pmod{27}. \quad (48)$$

By the binomial theorem, it is easy to see that for positive integers k and m ,

$$f_{3k}^{3m} \equiv f_k^{9m} \pmod{9}. \quad (49)$$

Invoking (49) in (48), we see that

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(4n+1)q^n \equiv 3 \frac{f_1^{12} f_2^3 f_3^2}{f_6} \pmod{27}. \quad (50)$$

Substituting (35) into (50), we obtain

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(4n+1)q^n \equiv 3 \frac{f_2^5 f_8^4 f_{12}^8}{f_4^4 f_6^3 f_{24}^4} + 3q^2 \frac{f_2 f_4^8 f_6 f_{24}^4}{f_8^4 f_{12}^4} - 6q \frac{f_2^3 f_4^2 f_{12}^2}{f_6} \pmod{27}. \quad (51)$$

Extracting the terms involving q^{2n+1} from (51), dividing by q and replacing q^2 by q , we get

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(8n+5)q^n \equiv 21 \frac{f_1^3 f_2^2 f_6^2}{f_3} \pmod{27}. \quad (52)$$

Employing (33) into (52), the equation reduces to

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(8n+5)q^n \equiv 21 \frac{f_2^2 f_4^3 f_6^2}{f_{12}} + 18q \frac{f_2^4 f_{12}^2}{f_4} \pmod{27},$$

which implies

$$\sum_{n=0}^{\infty} \bar{C}_{1,2}^6(16n+5)q^n \equiv 21 \frac{f_1^2 f_2^3 f_3^2}{f_6} \pmod{27}. \quad (53)$$

In view of congruences (50) and (53), we see that

$$\bar{C}_{1,2}^6(16n+5) \equiv 7 \cdot \bar{C}_{1,2}^6(4n+1) \pmod{27}.$$

Using the above relation and by induction on α , we arrive at (6).

4 Proof of Theorem 2

Setting $i = 1$, $j = 5$ and $\delta = 12$ in (5), we find that

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(n)q^n = \frac{f(q, q^{11})f(q^5, q^7)}{(q; q)_{\infty}^2} = \frac{f_2^2 f_3 f_{12}^3}{f_1^3 f_4 f_6^2}. \quad (54)$$

By the binomial theorem, it is easy to see that for positive integers k and m ,

$$f_{2k}^{2m} \equiv f_k^{2m} \pmod{2}, \quad (55)$$

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod{4}. \quad (56)$$

Invoking (56) in (54), we see that

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(n)q^n \equiv \frac{f_1 f_3 f_{12}^3}{f_4 f_6^2} \pmod{4}. \quad (57)$$

Substituting (39) into (57), we obtain

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(n)q^n \equiv \frac{f_3 f_9 f_{18}}{f_6} - q \frac{f_3^2 f_{18}^4}{f_6^2 f_9^2} - q^2 \frac{f_3 f_9 f_{36}^3}{f_{12} f_{18}^2} \pmod{4}.$$

Extracting the terms involving q^{3n} , q^{3n+1} and q^{3n+2} from the above equation, we obtain respectively (7), (8) and (9).

In view of congruences (9) and (57), we deduce that

$$\bar{C}_{1,5}^{12}(3n+2) \equiv 3 \cdot \bar{C}_{1,5}^{12}(n) \pmod{4}.$$

Using the above relation and by induction on α , we arrive at (10).

5 Proof of Theorem 3

Substituting (35) in (7), we find that

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(3n)q^n \equiv \frac{f_8^2 f_{12}^4}{f_4^2 f_{24}^2} - q \frac{f_4^4 f_6^2 f_{24}^2}{f_2^2 f_8^2 f_{12}^2} \pmod{4}. \quad (58)$$

Collecting the terms involving q^{2n} from (58) and replacing q^2 by q , we get

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(6n)q^n \equiv \frac{f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{4}. \quad (59)$$

Using (56) in (59), we have

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(6n)q^n \equiv f_2^2 \pmod{4}. \quad (60)$$

Extracting the terms involving q^{2n+1} from (60), we obtain (11).

Extracting the terms involving q^{2n+1} from (58), dividing by q and replacing q^2 by q , we arrive at

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(6n+3)q^n \equiv 3 \frac{f_2^4 f_3^2 f_{12}^2}{f_1^2 f_4^2 f_6^2} \pmod{4}. \quad (61)$$

Invoking (56) in (61), we get

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(6n+3)q^n \equiv 3 \frac{f_3^2 f_6^2}{f_1^2} \pmod{4}. \quad (62)$$

Employing (36) into (62), we have

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(6n+3)q^n \equiv 3 \frac{f_4^4 f_6^3 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^4 f_8 f_{24}}{f_2^4 f_{12}} \pmod{4}.$$

Collecting the even term of the above equation, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+3)q^n \equiv 3 \frac{f_2^4 f_3^3 f_6^2}{f_1^5 f_4 f_{12}} \pmod{4}. \quad (63)$$

Using (56) in (63), we see that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+3)q^n \equiv 3 \frac{f_2^2 f_3^3 f_6^2}{f_1 f_4 f_{12}} \pmod{4}. \quad (64)$$

Substituting (30) into (64), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+3)q^n \equiv 3 \frac{f_4^2 f_6^4}{f_{12}^2} + 3q \frac{f_2^2 f_6^2 f_{12}^2}{f_4^2} \pmod{4}, \quad (65)$$

which implies that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(24n+3)q^n \equiv 3 \frac{f_2^2 f_3^4}{f_6^2} \pmod{4}. \quad (66)$$

Invoking (56) in (66), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(24n+3)q^n \equiv 3 f_2^2 \pmod{4}. \quad (67)$$

Extracting the terms involving q^{2n+1} from the above equation, we get (12).

Extracting the terms involving q^{2n+1} from (65), dividing by q and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(24n+15)q^n \equiv 3 \frac{f_1^2 f_3^2 f_6^2}{f_2^2} \pmod{4}. \quad (68)$$

Using (56) in (68), we see that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(24n+15)q^n \equiv 3 \frac{f_3^2 f_6^2}{f_1^2} \pmod{4}. \quad (69)$$

Employing (36) into (69), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(24n+15)q^n \equiv 3 \frac{f_4^4 f_6^3 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^4 f_8 f_{24}}{f_2^4 f_{12}} \pmod{4}, \quad (70)$$

which implies,

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(48n+15)q^n \equiv 3 \frac{f_2^4 f_3^3 f_6^2}{f_1^5 f_4 f_{12}} \pmod{4}. \quad (71)$$

Invoking (56) in (71), we get

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(48n+15)q^n \equiv 3 \frac{f_2^2 f_3^3 f_6^2}{f_1 f_4 f_{12}} \pmod{4}. \quad (72)$$

In view of congruences (64) and (72), we obtain

$$\overline{C}_{1,5}^{12}(48n+15) \equiv \overline{C}_{1,5}^{12}(12n+3) \pmod{4}.$$

Using the above relation and by induction on α , we arrive at

$$\overline{C}_{1,5}^{12}(3 \cdot 4^{\alpha+2}n + 4 \cdot 4^{\alpha+2} - 1) \equiv \overline{C}_{1,5}^{12}(12n+3) \pmod{4}. \quad (73)$$

Using the congruence (12) in (73), we obtain (14).

Extracting the terms involving q^{2n+1} from (70), dividing by q and replacing q^2 by q , we have

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12} (48n+39) q^n \equiv 2 \frac{f_2 f_3^4 f_4 f_{12}}{f_1^4 f_6} \pmod{4}. \quad (74)$$

Invoking (55) in (74), we get

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12} (48n+39) q^n \equiv 2 f_2 f_6 f_{12} \pmod{4}. \quad (75)$$

Congruence (13) follows by extracting the terms involving q^{2n+1} from the above equation.

6 Proof of Theorem 4

Employing (37) into (8), we have

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12} (3n+1) q^n \equiv 3 \frac{f_4^2 f_{12}^4}{f_2 f_6 f_8 f_{24}} + 2q \frac{f_6^3 f_8 f_{12} f_{24}}{f_4} \pmod{4}. \quad (76)$$

Extracting the terms involving q^{2n} from (76) and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12} (6n+1) q^n \equiv 3 \frac{f_2^2 f_6^4}{f_1 f_3 f_4 f_{12}} \pmod{4}. \quad (77)$$

Substituting (34) into (77), we arrive at

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12} (6n+1) q^n \equiv 3 \frac{f_8^2 f_{12}^4}{f_4^2 f_{24}^2} + 3q \frac{f_4^4 f_6^2 f_{24}^2}{f_2^2 f_8^2 f_{12}^2} \pmod{4}. \quad (78)$$

Extracting the terms involving q^{2n} from (78) and replacing q^2 by q , we deduce that

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12} (12n+1) q^n \equiv 3 \frac{f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{4}. \quad (79)$$

Invoking (56) in (79), we get

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12} (12n+1) q^n \equiv 3 f_2^2 \pmod{4}. \quad (80)$$

Extracting the terms involving q^{2n+1} from (80), we obtain (15) and combining (67) and (80), we get (16).

From the (78), which implies that

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12} (12n+7) q^n \equiv 3 \frac{f_2^4 f_3^2 f_{12}^2}{f_1^2 f_4^2 f_6^2} \pmod{4}. \quad (81)$$

Using (56) in (81), we have

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12} (12n+7) q^n \equiv 3 \frac{f_3^2 f_6^2}{f_1^2} \pmod{4}. \quad (82)$$

Combining (69) and (82), we arrive at (17).

7 Proof of Theorem 5

From (60), we have

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(12n)q^n \equiv f_1^2 \pmod{4}.$$

Employing (29) in the above equation and then extracting the terms containing q^{5n+2} , dividing by q^2 and replacing q^5 by q , we get

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(60n+24)q^n \equiv f_5^2 \pmod{4}, \quad (83)$$

which yields

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(300n+24)q^n \equiv f_1^2 \equiv \sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(12n)q^n \pmod{4}. \quad (84)$$

By induction on α , we obtain (18).

The congruence (19) follows by extracting the terms involving q^{5n+i} for $i = 1, 2, 3, 4$ from both sides of (83).

8 Proof of Theorem 6

Extracting the terms involving q^{2n} from (75) and replacing q^2 by q we have

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(96n+39)q^n \equiv 2f_1f_3f_6 \pmod{4}. \quad (85)$$

Substituting (35) into (85), we arrive at

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(96n+39)q^n \equiv 2 \frac{f_2f_8^2f_{12}^4}{f_4^2f_{24}^2} - 2q \frac{f_4^4f_6^2f_{24}^2}{f_2f_8^2f_{12}^2} \pmod{4}.$$

Extracting the even terms in the above equation, we obtain

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(192n+39)q^n \equiv 2 \frac{f_1f_4^2f_6^4}{f_2^2f_{12}^2} \pmod{4}. \quad (86)$$

Using (55) in (86), we see that

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(192n+39)q^n \equiv 2f_1f_4 \pmod{4}. \quad (87)$$

Define

$$\sum_{n=0}^{\infty} f(n)q^n = f_1f_4. \quad (88)$$

Combining (87) and (88), we find that

$$\sum_{n=0}^{\infty} \bar{C}_{1,5}^{12}(192n+39)q^n \equiv 2 \sum_{n=0}^{\infty} f(n)q^n \pmod{4}. \quad (89)$$

For a prime, $p \geq 5$ or $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$, consider

$$\frac{3k^2 + k}{2} + 4 \cdot \frac{3m^2 + m}{2} \equiv \frac{5p^2 - 5}{24} \pmod{p}, \quad (90)$$

therefore,

$$(6k + 1)^2 + 4 \cdot (6m + 1)^2 \equiv 0 \pmod{p},$$

Since $\left(\frac{-4}{p}\right) = -1$ the congruence relation (90) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute Lemma 9 into (88) and then extract the terms in which the powers of q are congruent to $\frac{5p^2-5}{24}$ modulo p and then divide by $q^{\frac{5p^2-5}{24}}$, we find that

$$\sum_{n=0}^{\infty} f\left(pn + \frac{5p^2 - 5}{24}\right) q^{pn} = f_{p^2} f_{4p^2},$$

which implies that

$$\sum_{n=0}^{\infty} f\left(p^2n + \frac{5p^2 - 5}{24}\right) q^n = f_1 f_4$$

and for $n \geq 0$,

$$f\left(p^2n + pi + \frac{5p^2 - 5}{24}\right) = 0, \quad (91)$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha}n + \frac{5p^{2\alpha} - 5}{24}\right) = f(n). \quad (92)$$

Replacing n by $p^{2\alpha}n + \frac{5p^{2\alpha}-5}{24}$ in (89), we arrive at (20).

9 Proof of Theorem 7

Replacing n by $p^2n + pi + \frac{5p^2-5}{24}$ in (92) and using (91), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{5p^{\alpha+2} - 5}{24}\right) = 0. \quad (93)$$

Comparing coefficients of q^n from both sides of (89), we see that for $n \geq 0$,

$$\bar{C}_{1,5}^{12}(192n + 39) \equiv 2f(n) \pmod{4}. \quad (94)$$

The required result follows from (93) and (94).

10 Proof of Theorem 8

Setting $i = 3$, $j = 3$ and $\delta = 9$ in (5), we have

$$\sum_{n=0}^{\infty} \bar{C}_{3,3}^9(n) q^n = \frac{f(q^3, q^6) f(q^3, q^6)}{(q; q)_{\infty}^2}.$$

After q -product manipulation, we see that

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^9(n) q^n = \frac{f_6^2 f_9^4}{f_1^2 f_3^2 f_{18}^2}. \quad (95)$$

Invoking (56) in (95), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^9(n) q^n \equiv \frac{f_3^2}{f_1^2} \pmod{4}. \quad (96)$$

Employing (36) into (96), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^9(n) q^n \equiv \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \pmod{4}.$$

Extracting the odd terms of the above equation, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^9(2n+1) q^n \equiv 2 \frac{f_2 f_3^2 f_4 f_{12}}{f_1^4 f_6} \pmod{4}. \quad (97)$$

Using (55) in (97), we get

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^9(2n+1) q^n \equiv 2 f_2 f_{12} \pmod{4}, \quad (98)$$

which implies,

$$\overline{C}_{3,3}^9(4n+3) \equiv 0 \pmod{4}.$$

Extracting the even terms from (98), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^9(4n+1) q^n \equiv 2 f_1 f_6 \pmod{4}. \quad (99)$$

Define

$$\sum_{n=0}^{\infty} g(n) q^n = f_1 f_6. \quad (100)$$

Combining (99) and (100), we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^9(4n+1) q^n \equiv 2 \sum_{n=0}^{\infty} g(n) q^n \pmod{4}. \quad (101)$$

For a prime, $p \geq 5$ or $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$, consider

$$\frac{3k^2 + k}{2} + 6 \cdot \frac{3m^2 + m}{2} \equiv \frac{7p^2 - 7}{24} \pmod{p}, \quad (102)$$

therefore,

$$(6k+1)^2 + 6 \cdot (6m+1)^2 \equiv 0 \pmod{p},$$

Since $\left(\frac{-6}{p}\right) = -1$ the congruence relation (102) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute Lemma 9 into (100) and then extract the terms in which the powers of q are congruent to $\frac{7p^2-7}{24}$ modulo p and then divide by $q^{\frac{7p^2-7}{24}}$, we find that

$$\sum_{n=0}^{\infty} g\left(pn + \frac{7p^2-7}{24}\right) q^{pn} = f_{p^2} f_{6p^2},$$

which implies that

$$\sum_{n=0}^{\infty} g\left(p^2n + \frac{7p^2-7}{24}\right) q^n = f_1 f_6$$

and for $n \geq 0$,

$$g\left(p^2n + pi + \frac{7p^2-7}{24}\right) = 0, \quad (103)$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$g\left(p^{2\alpha}n + \frac{7p^{2\alpha}-7}{24}\right) = g(n). \quad (104)$$

Replacing n by $p^{2\alpha}n + \frac{7p^{2\alpha}-7}{24}$ in (101), we arrive at (21).

11 Proof of Theorem 9

Replacing n by $p^2n + pi + \frac{7p^2-7}{24}$ in (104) and using (103), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$g\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{7p^{\alpha+2}-7}{24}\right) = 0. \quad (105)$$

Comparing coefficients of q^n from both sides of (101), we see that for $n \geq 0$,

$$\overline{C}_{3,3}^9(4n+1) \equiv 2g(n) \pmod{4}. \quad (106)$$

The required result follows from (105) and (106).

12 Proof of Theorem 10

Putting $i = 5$, $j = 5$ and $\delta = 15$ in (5), we find that

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(n) q^n = \frac{f(q^5, q^{10})^2}{(q; q)_{\infty}^2}.$$

After q -product manipulation, we see that

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(n) q^n = \frac{f_{10}^2 f_{15}^4}{f_1^2 f_5^2 f_{30}^2}. \quad (107)$$

Invoking (56) in (107), we have

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(n) q^n \equiv \frac{f_5^2}{f_1^2} \pmod{4}. \quad (108)$$

Substituting (42) into (108), we get

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(n) q^n \equiv \left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \right)^2 \pmod{4},$$

which implies that

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(2n+1)q^n \equiv 2f_1f_5^3 \pmod{4}. \quad (109)$$

Employing (43) into (109), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(2n+1)q^n \equiv 2f_2^3f_{10} + 2q \frac{f_2^2f_{10}^2f_{20}}{f_4} \pmod{4}. \quad (110)$$

Extracting the terms involving q^{2n} from (110) and replacing q^2 by q , we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(4n+1)q^n \equiv 2f_1^3f_5 \pmod{4} \quad (111)$$

Substituting (44) into (111), the equation reduces to

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(4n+1)q^n \equiv 2 \frac{f_2^2f_4f_{10}^2}{f_{20}} + 2qf_2f_{10}^3 \pmod{4}, \quad (112)$$

which implies that

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(8n+5)q^n \equiv 2f_1f_5^3 \pmod{4}. \quad (113)$$

Combining (109) and (113), we get

$$\overline{C}_{5,5}^{15}(8n+5) \equiv \overline{C}_{5,5}^{15}(2n+1) \pmod{4}.$$

Using the above relation and by induction on α , we arrive at (26).

Extracting the terms involving q^{2n+1} from (110), dividing by q and replacing q^2 by q , we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(4n+3)q^n \equiv 2 \frac{f_1^2f_5^2f_{10}}{f_2} \pmod{4}.$$

Using (55), one can reduce the above equation to

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(4n+3)q^n \equiv 2f_5^4 \pmod{4}. \quad (114)$$

Collecting the terms involving q^{5n+i} on both sides of (114), we find that

$$\overline{C}_{5,5}^{15}(4(5n+i)+3)q^n \equiv 0 \pmod{4}, \quad i = 1, 2, 3, 4. \quad (115)$$

Extracting the terms involving q^{5n} from (114), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(20n+3)q^n \equiv 2f_1^4 \pmod{4}. \quad (116)$$

Employing (29) into (116) and extracting the terms involving q^{5n+4} in the resultant equation, we have

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(100n+83)q^n \equiv 2f_5^4 \pmod{4}. \quad (117)$$

In the view of congruences (114) and (117), we obtain

$$\overline{C}_{5,5}^{15}(100n+83)q^n \equiv \overline{C}_{5,5}^{15}(4n+3) \pmod{4}.$$

Using the above relation and by induction on α , we have

$$\overline{C}_{5,5}^{15} \left(4 \cdot 5^{5\alpha+5} n + \frac{2 \cdot 5^{5\alpha+6} - 1}{3} \right) \equiv \overline{C}_{5,5}^{15} (4n + 3) \pmod{4}.$$

Using congruence (115) in the above equation, we get (27).

Extracting the even terms of (112), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15} (8n + 1) q^n \equiv 2 \frac{f_1^2 f_2 f_5^2}{f_{10}} \pmod{4}. \quad (118)$$

Invoking (55) in (118), we deduce

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15} (8n + 1) q^n \equiv 2 f_2^2 \pmod{4}. \quad (119)$$

Congruence (23) follows by extracting the terms involving q^{2n+1} from the above equation.

Using (55) in (119) implies

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15} (8n + 1) q^n \equiv 2 f_1^4 \pmod{4}. \quad (120)$$

Combining (116) and (120), we obtain (24).

Substituting (29) into (109), we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15} (2n + 1) q^n \equiv 2 f_5^3 f_{25} \left(a(q^5) - q - \frac{q^2}{a(q^5)} \right) \pmod{4}. \quad (121)$$

Extracting the terms involving q^{5n+j} on both sides of (121), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15} (2(5n + j) + 1) \equiv 0 \pmod{4}, \quad j = 3, 4. \quad (122)$$

Extracting the terms involving q^{5n+1} from (121), dividing by q and replacing q^5 by q , we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15} (10n + 3) q^n \equiv 2 f_1^3 f_5 \pmod{4}. \quad (123)$$

From (111) and (123), we obtain (25).

Employing (29) into (123), we arrive at

$$\begin{aligned} & \sum_{n=0}^{\infty} \overline{C}_{5,5}^{15} (10n + 3) q^n \\ & \equiv f_5 f_{25}^3 \left(a^3(q^5) - 3a^2(q^5)q + 5q^3 - \frac{3q^5}{a^2(q^5)} - \frac{q^6}{a^3(q^5)} \right) \pmod{4}, \end{aligned} \quad (124)$$

which implies that

$$\overline{C}_{5,5}^{15} (10(5n + k) + 3) \equiv 0 \pmod{4}, \quad k = 2, 4.$$

Extracting the terms involving q^{5n+3} from (124), dividing by q^3 and replacing q^5 by q , we have

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(50n+33)q^n \equiv 2f_1 f_5^3 \pmod{4}. \quad (125)$$

Combining (109) and (125), we get

$$\overline{C}_{5,5}^{15}(50n+33) \equiv \overline{C}_{5,5}^{15}(2n+1) \pmod{4}.$$

Using the above relation and by induction on α , we have

$$\overline{C}_{5,5}^{15} \left(2 \cdot 5^{2\alpha+4} n + \frac{4 \cdot 5^{2\alpha+4} - 1}{3} \right) \equiv \overline{C}_{5,5}^{15}(2n+1) \pmod{4}. \quad (126)$$

Using (122) in (126), we obtain (28).

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