UNIQUENESS OF DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS
Harina P. Waghmore\textsuperscript{1 \S}, Rajeshwari S.\textsuperscript{2}
\textsuperscript{1,2}Department of Mathematics
Jnanabharathi Campus
Bangalore University
Bangalore, 560 056, INDIA

Abstract: In this paper, we study the zero distributions on the derivatives of q-shift difference polynomials of meromorphic functions with zero order and obtain two theorems that extend results of [3].

AMS Subject Classification: 30D35
Key Words: uniqueness, meromorphic function, difference polynomials, shared function

1. Introduction

In this paper, a meromorphic functions $f$ means meromorphic in the complex plane. If no poles occur, then $f$ reduces to an entire function. Throughout of this paper, we denote by $\rho(f)$ and $\rho_2(f)$ the order of $f$ and the hyper order of $f$ (Laine, 1993 and Yang and Yi, 2003). In addition, if $f - a$ and $g - a$ have the same zeros, then we say that $f$ and $g$ share the value $a$ IM(ignoring multiplicities). If $f - a$ and $g - a$ have the same zeros, then we say that $f$ and $g$ share the value $a$ CM(counting multiplicities). We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory(Halburd Korhonen and Tohge; Laine, 1993 and Yang and Yi, 2003).

Given a meromorphic function $f(z)$, recall that $\alpha(z) \neq 0, \infty$ is a small function with respect to $f(z)$, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ is used to denote any quantity satisfying $S(r, f) = o(T(r, f))$, and $r \rightarrow \infty$ outside of a
possible exceptional set of finite logarithmic measure.

Recently, K. Liu, X. Liu and T. B. Cao (2012) proved the following.

**Theorem A.** (Liu, Liu and Coa, 2012) Let \( f \) be a transcendental entire function of \( \rho_2(f) < 1 \). For \( n \geq t(k + 1) + 1 \), then \([P(f)f(z + c)]^{(k)} - \alpha(z)\) has infinitely many zeros.

**Theorem B.** (Liu, Liu and Coa, 2012) Let \( f \) be a transcendental meromorphic function of \( \rho_2(f) < 1 \), not a periodic function with period \( c \). If \( n \geq (t+1)(k+1) + 1 \), then \([f(z)^n(\Delta_c f)^s]^{(k)} - \alpha(z)\) has infinitely many zeros.

**Theorem C.** (Liu, Liu and Coa, 2012) Let \( f \) be a transcendental meromorphic function of \( \rho_2(f) < 1 \). For \( n \geq t(k + 1) + 5 \), then \([P(f)f(z + c)]^{(k)} - \alpha(z)\) has infinitely many zeros.

**Theorem D.** (Liu, Liu and Coa, 2012) Let \( f \) be a transcendental meromorphic function of \( \rho_2(f) < 1 \). If \( n \geq (t+2)(k+1) + 3 + s \), then \([P(f)(\Delta_c f)^s]^{(k)} - \alpha(z)\) has infinitely many zeros.

**Theorem E.** (Liu, Liu and Coa, 2012) Let \( f \) and \( g \) be a transcendental entire function of \( \rho_2(f) < 1 \), \( n \geq 2k + m + 6 \). If \([f^n(f^m-1)f(z + c)]^{(k)}\) and \([g^n(g^m-1)g(z + c)]^{(k)}\) share the 1 CM, then \( f = tg \), where \( t^{n+1} = t^m = 1 \).

**Theorem F.** (Liu, Liu and Coa, 2012) The conclusion of Theorem E is also valid, if \( n \geq 5k + 4m + 12 \). and \([f^n(f^m-1)f(z + c)]^{(k)}\) and \([g^n(g^m-1)g(z + c)]^{(k)}\) share the 1 IM.

In 2013, Harina P. Waghamore and Tanuja A. extend Theorem E and Theorem F to meromorphic functions.

**Theorem G.** (Harina P.W and Tanuja A, 2013) Let \( f \) and \( g \) be a transcendental meromorphic function with zero order. If \( n \geq 4k + m + 8 \), \([f^n(f^m-1)f(qz + c)]^{(k)}\) and \([g^n(g^m-1)g(qz + c)]^{(k)}\) share the 1 CM, then \( f = tg \), where \( t^{n+1} = t^m = 1 \).

**Theorem H.** (Harina P.W and Tanuja A, 2013) Let \( f \) and \( g \) be a transcendental meromorphic function with zero order. If \( n \geq 5k + 4m + 17 \), \([f^n(f^m-1)f(qz + c)]^{(k)}\) and \([g^n(g^m-1)g(qz + c)]^{(k)}\) share the 1 IM, then \( f = tg \), where \( t^{n+1} = t^m = 1 \).
In this paper, we extend Theorem G and Theorem H to difference polynomials and obtain the following results.

**Theorem 1.** Let \( f \) and \( g \) be a transcendental meromorphic (resp. entire) function with zero order. If \( n \geq 4k + 8(n \geq 2k + 6) \), \([P(f)f(qz + c)]^{(k)}\) and \([P(g)g(qz + c)]^{(k)}\) share the 1 CM, then:

1. \( f \equiv t g \) for a constant \( t \) such that \( t^d = 1 \).
2. \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) \equiv 0 \), where \( R(w_1, w_2) = P(w_1)w_1(qz + c) - P(w_2)w_2(qz + c) \).

**Theorem 2.** Let \( f \) and \( g \) be a transcendental meromorphic (resp. entire) function with zero order. If \( n \geq 10k + 14(n \geq 5k + 12) \), \([P(f)f(qz + c)]^{(k)}\) and \([P(g)g(qz + c)]^{(k)}\) share the 1 IM, then the conclusion of theorem 1 still holds.

### 2. Some Lemmas

In this section, we present some definitions and lemmas which will be needed in the sequel.

**Lemma 2.1.** (Halburd, Korhonen and Tohge, Theorem 5.1) Let \( f(z) \) be a transcendental meromorphic function of \( \rho_1(f) < 1, \zeta < 1, \epsilon \) is enough small number. Then

\[
m(r, \frac{f(z + c)}{f(z)}) = o \left( \frac{T(r, f)}{r^{1-\epsilon}} \right) = S(r, f),
\]

for all \( r \) outside of a set of finite logarithmic measure. Combining the proof of (Luo and Lin, 2011, Lemma 5) with Lemma 2.1, we can get the following Lemma 2.2.

**Lemma 2.2.** Let \( f(z) \) be a transcendental entire function of \( \rho_2(f) < 1 \). If \( F = P(f)f(z + c) \), then

\[
T(r, F) = T(P(f)f(z)) + S(r, f) = (n + 1)T(r, f) + S(r, f).
\]

**Lemma 2.3.** (Liu, Liu and Cao, 2012, Lemma 2.5) Let \( f(z) \) be a transcendental meromorphic function of \( \rho_2(f) < 1 \). If \( F = P(f)f(z + c) \), then

\[
(n - 1)T(r, f) + S(r, f) \leq T(r, F) \leq (n + 1)T(r, f) + S(r, f).
\]
Lemma 2.4. (Zhang and Korhonen, 2010, Theorem 1.1) Let $f(z)$ be a transcendental meromorphic function of zero order. Then

$$T(r, f(qz)) = T(r, f(z)) + S(r, f)$$

on a set of logarithmic density 1.

The following lemma has little modifications of the original version (Theorem 2.1 of Chiang and Feng, 2008).

Lemma 2.5. Let $f(z)$ be a transcendental meromorphic function of finite order. Then

$$T(r, f(z + c)) = T(r, f) + S(r, f). \quad (2.4)$$

combining Lemma 2.4 with Lemma 2.5, we get the following result easily.

Lemma 2.6. Let $f(z)$ be a transcendental meromorphic function of zero order. Then

$$T(r, f(qz + c)) = T(r, f(z)) + S(r, f) \quad (2.5)$$

on a set of logarithmic density 1.

Lemma 2.7. (Yang and Hua, 1997, Lemma 3) Let $F$ and $G$ be non constant meromorphic functions. If $F$ and $G$ share 1 CM, then one of the following three cases holds:

(i) $\max \{T(r, F), T(r, G)\} \leq N_2 \left( r, \frac{1}{F} \right) + N_2 (r, F) + N_2 \left( r, \frac{1}{G} \right) + N_2 (r, G) + S(r, F) + S(r, G)$.

(ii) $F = G$.

(iii) $F.G = 1$.

Lemma 2.8. (Xu an Yi, 2007, Lemma 2.3) Let $F$ and $G$ be non constant meromorphic function sharing the value 1 IM. Let

$$H = \frac{F''}{F'} - 2 \frac{F'}{F - 1} - \frac{G''}{G'} + 2 \frac{G'}{G - 1}.$$ 

If $H \neq 0$, then

$$T(r, F) + T(r, G) \leq 2 \left( N_2 \left( r, \frac{1}{F} \right) + N_2 (r, F) + N_2 \left( r, \frac{1}{G} \right) + N_2 (r, G) \right)$$

$$+ S(r, F) + S(r, G).$$
\[ + 3 \left( \overline{N}(r, F) + \overline{N} \left( r, \frac{1}{F} \right) + \overline{N}(r, G) + \overline{N} \left( r, \frac{1}{G} \right) \right) + S(r, F) + S(r, G). \]  

(2.6)

**Lemma 2.9.** Let \( f(z) \) be a meromorphic function, and \( p, k \) be positive integers. Then

\[ T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + S(r, f). \]  

(2.7)

\[ N_p \left( r, \frac{1}{f^{(k)}} \right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k} \left( r, \frac{1}{f} \right) + S(r, f). \]  

(2.8)

\[ N_p \left( r, \frac{1}{f^{(k)}} \right) \leq k\overline{N}(r, f) + N_{p+k} \left( r, \frac{1}{f} \right) + S(r, f). \]  

(2.9)

**Lemma 2.10.** Let \( f \) and \( g \) be a transcendental meromorphic function of zero order. If \( n \geq k + 6 \) and

\[ [P(f)f(qz + c)]^{(k)} = [P(g)g(qz + c)]^{(k)} \]  

(2.10)

then \( f = tg \), where \( t^{n+1} = t^m = 1 \), and \( f \) and \( g \) satisfy the algebraic equation

\[ R(w_1, w_2) = P(w_1)w_1(qz + c) - P(w_2)w_2(qz + c). \]

**Proof.** From (2.10), we have

\[ P(f)f(qz + c) = P(g)g(qz + c) + Q(z). \]

Where \( Q(z) \) is a polynomial of degree atmost \( k = 1 \). If \( Q(z) \neq 0 \), then we have

\[ \frac{P(f)f(z + c)}{Q(z)} = \frac{P(g)g(qz + c)}{Q(z)} + 1 \]
From the second main theorem of Nevanlinna and by Lemma 2.2, we have

\[(n + 1)T(r, f) = T\left(r, \frac{P(f)f(qz + c)}{Q(z)}\right) + S(r, f)\]
\[\leq \overline{N}\left(r, \frac{P(f)f(qz + c)}{Q(z)}\right) + \overline{N}\left(r, \frac{Q(z)}{P(f)f(qz + c)}\right)\]
\[+ \overline{N}\left(r, \frac{Q(z)}{P(g)g(qz + c)}\right) + S(r, f)\]
\[\leq \overline{N}(r, P(f)) + \overline{N}(r, f(qz + c)) + \overline{N}\left(r, \frac{1}{P(f)}\right)\]
\[+ \overline{N}\left(r, \frac{1}{f(qz + c)}\right)\]
\[+ \overline{N}\left(r, \frac{1}{g(z)}\right) + \overline{N}\left(r, \frac{1}{g(qz + c)}\right) + S(r, f) + S(r, g)\]
\[\leq 4T(r, f) + 2T(r, g) + S(r, f) + S(r, g).\]  \hspace{1cm} (2.11)

Similarly as above, we have

\[(n + 1)T(r, g) \leq 4T(r, g) + 2T(r, f) + S(r, f) + S(r, g).\]  \hspace{1cm} (2.12)

Thus, we get

\[(n + 1)[T(r, f) + T(r, g)] \leq 6[T(r, f) + T(r, g)] + S(r, f) + S(r, g).\]  \hspace{1cm} (2.13)

which is in contradiction with \(n \geq k + 6\). Hence, we get \(Q(z) \equiv 0\), which implies that

\[P(f)f(qz + c) = P(g)g(qz + c).\]  \hspace{1cm} (2.14)

Set \(h(z) = \frac{f(z)}{g(z)}\), we break the rest of the proof into two cases.

**Case 1.** Suppose \(h(z)\) is a constant. Then by substituting \(f = gh\) into (2.14), we obtain

\[g(qz + c)[a_ng^n(h^{n+1} - 1) + a_{n-1}g^{n-1}(h^n - 1) + ... + a_0(h - 1)] \equiv 0\]  \hspace{1cm} (2.15)

where \(a_n(\neq 0), a_{n-1}, ..., a_0\) are complex constants. By the fact that \(g\) is a transcendental entire functions, we have \(g(qz + c) \neq 0\). Hence, we obtain

\[a_ng^n(h^{n+1} - 1) + a_{n-1}g^{n-1}(h^n - 1) + ... + a_0(h - 1) \equiv 0.\]  \hspace{1cm} (2.16)

Equation (2.16) implies that \(h^{n+1} = 1\) and \(h^{i+1} = 1\) when \(a_i \neq 0\) for \(i = 0, 1, ..., n - 1\). Therefore \(h^d = 1\), where \(d = \text{GCD}(\lambda_0, \lambda_1, ..., \lambda_n)\).
Case 2. Suppose that \( h \) is not a constant, then we know by (2.14) that \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) = 0 \), where \( R(w_1, w_2) = p(w_1)w_1(qz + c) - p(w_2)w_2(qz + c) \).

Lemma 2.11. Let \( f \) and \( g \) be transcendental entire function of finite order. If \( n \geq k + 4 \), and \( [P(f)g(qz + c)]^{(k)} = [P(g)g(qz + c)]^{(k)} \) then the condition of Lemma 2.10 holds.

Proof. Substituting \( N(r, f) = N(r, g) = 0 \) and proceeding as in the proof of Lemma 2.10, we get Lemma 2.11.

3. Proof of the Theorem

Proof of Theorem 1.1. Let \( F = [P(f)g(qz + c)]^{(k)} \) and \( G = [P(g)g(qz + c)]^{(k)} \). Thus \( F \) and \( G \) share the value 1 CM. From (2.7) and \( f \) is a transcendental meromorphic function, then

\[
T(r, F) \leq T(r, P(f)f(qz + c)) + kN(r, f) + S(r, P(f)f(qz + c)) \tag{3.1}
\]

combining (3.1) with Lemma 2.2, we have \( S(r, F) = S(r, f) \). We also have \( S(r, G) = S(r, g) \), from the same reason as above, from (2.8) we obtain

\[
N_2(r, \frac{1}{F}) = N_2 \left( r, \frac{1}{[P(f)f(qz + c)]^{(k)}} \right) \\
\leq T(r, F) - T(r, P(f)f(qz + c)) \\
+ N_{k+2} \left( r, \frac{1}{P(f)f(qz + c)} \right) + S(r, f). \tag{3.2}
\]

Thus, from Lemma 2.2 and (3.2) we get

\[
(n + 1)T(r, f) = T(r, P(f)f(qz + c)) + S(r, f) \\
\leq T(r, F) - N_2(r, \frac{1}{F}) \\
+ N_{k+2} \left( r, \frac{1}{P(f)f(qz + c)} \right) + S(r, f) \tag{3.3}
\]

From (2.9), we obtain

\[
N_2(r, \frac{1}{F}) \leq N_{k+2} \left( r, \frac{1}{P(f)f(qz + c)} \right) + S(r, f) \\
\leq (k + 2)N(r, \frac{1}{f}) + N \left( r, \frac{1}{f(qz + c)} \right) + k\overline{N}(r, f) + S(r, f) \tag{3.4}
\]

\[
\leq (2k + 3)T(r, f) + S(r, f).
\]
Similarly as above, we have

\((n + 1)T(r, g) \leq T(r, G) - N_2(r, \frac{1}{G}) + N_{k+2} \left( r, \frac{1}{P(g)g(qz + c)} \right) + S(r, g) \) \hspace{1cm} (3.5)

\[N_2(r, \frac{1}{G}) \leq (2k + 3)T(r, g) + S(r, g).\] \hspace{1cm} (3.6)

If the (i) of Lemma 2.7 is satisfied implies that

\[\max \{T(r, F)T(r, G)\} \leq N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + N_2(r, F) + N_2(r, G) + N_2(r, F) + S(r, F) + S(r, G).\]

Thus, combining above with (3.3)-(3.6) we obtain

\[(n + 1)\{T(r, f) + T(r, g)\} \leq 2[N(r, f) + N(r, g)] + 2N_{k+2} \left( r, \frac{1}{P(f)f(qz + c)} \right)
\[+ 2N_{k+2} \left( r, \frac{1}{P(g)g(qz + c)} \right) + S(r, f) + S(r, g) \leq 2(2k + 4)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).\]

Which is in contradiction with \(n \geq 4k + 8\). Hence \(F = G\) or \(FG = 1\). From Lemma 2.10, we get \(f = tg\) for \(t^m = t^{n+1} = 1\) and \(f\) and \(g\) satisfy the algebraic equation \(R(f, g) = 0\), where \(R(w_1, w_2) = P(w_1)w_1(qz + c) - p(w_2)w_2(qz + c)\).

**Proof of Theorem 1.2.** Let

\(F = [P(f)f(qz + c)]^{(k)}, \quad G = [P(g)g(qz + c)]^{(k)}\).

Let \(H\) be defined as in Lemma 2.8. Assume that \(H \neq 0\), from (2.6) we get

\[T(r, F) + T(r, G) \leq 2 \left[ N_2 \left( r, \frac{1}{F} \right) + N_2 \left( r, \frac{1}{G} \right) + N_2(r, F) + N_2(r, G) \right]
\[+ 3 \left[ N(r, F) + N(r, G) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) \right] \leq 2(2k + 4)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).\] \hspace{1cm} (3.7)
Combining above with (3.3)-(3.6) and (2.9), we obtain
\[(n + 1)[T(r, f) + T(r, g)] \leq T(r, F) + T(r, G) - N_2(r, \frac{1}{F}) - N_2(r, \frac{1}{G})\]
\[+ N_{k+2} \left( r, \frac{1}{P(f)f(qz + c)} \right) + N_{k+2} \left( r, \frac{1}{P(g)g(qz + c)} \right)\]
\[+ S(r, f) + S(r, g)\]
\[\leq 2(N_2(r, F) + N_2(r, G)) + 2N_{k+2} \left( r, \frac{1}{P(f)f(qz + c)} \right)\]
\[+ 2N_{k+2} \left( r, \frac{1}{P(g)g(qz + c)} \right) + 3 \left[ N(r, \frac{1}{F}) + N(r, \frac{1}{G}) \right]\]
\[+ S(r, f) + S(r, g)\]
\[\leq 2(2k + 4)\{T(r, f) + T(r, g)\} + 3(2k + 2)\{T(r, f) + T(r, g)\}\]
\[+ S(r, f) + S(r, g)\]
\[\leq (10k + 14)\{T(r, f) + T(r, g)\}\]

which is a contradiction with \( n \geq 10k + 14 \). Thus we get \( H \equiv 0 \). The following proof is trivial, we give the complete proof. By integration for \( H \) twice, we obtain
\[F = \frac{(b + 1)G + (a - b - 1)}{bG + (a - b)} \quad G = \frac{(a - b - 1) - (a - b)F}{Fb - (b + 1)} \quad (3.8)\]

which implies that \( T(r, F) = T(r, G) + O(1) \). Since
\[T(r, F) \leq T(r, P(f)f(qz + c)) + S(r, f)\]
\[\leq (n + 1)T(r, f) + S(r, f),\]

then \( S(r, F) = S(r, f) \). So \( S(r, G) = S(r, g) \) is. We distinguish into three cases as follows.

**Case 1.** \( b \neq 0, -1 \). If \( a - b - 1 \neq 0 \), then by (3.8), we get
\[\overline{N} \left( r, \frac{1}{F} \right) = \overline{N} \left( r, \frac{1}{F - \frac{a-b-1}{b+1}} \right) \quad (3.9)\]

By the Nevanlinna second main theorem, (2.8) and (2.9), we have
\[(n + 1)T(r, g) \leq T(r, G) + N_{k+2} \left( r, \frac{1}{P(g)g(qz + c)} \right)\]
\[- N \left( r, \frac{1}{G} \right) + S(r, g)\]
\[\leq (k + 1)T(r, g) + (2k + 2)T(r, f) + S(r, f) + S(r, g) \quad (3.10)\]
Similarly, we get
\[(n + 1)T(r, f) \leq (k + 1)T(r, f) + (2k + 2)T(r, g) + S(r, f) + S(r, g). \tag{3.11}\]
Thus from (3.10) and (3.11), then
\[(n + 1)\{T(r, f) + T(r, g)\} \leq (3k + 3)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g).
Which is in contradiction with \(n \geq 10k + 14\). Thus \(a - b - 1 = 0\), then
\[F = \frac{(b + 1)G}{bG + 1} \tag{3.12}\]
using the same method as above, we get
\[(n + 1)T(r, g) \leq T(r, G) + N_k \left( r, \frac{1}{P(g)g(qz + c)} \right) - N(r, \frac{1}{G}) + S(r, g)
\leq N_k \left( r, \frac{1}{P(g)g(qz + c)} \right) + N \left( r, \frac{1}{G + \frac{1}{b}} \right) + S(r, g)
\leq (k + 1)T(r, g) + S(r, g).
Which is a contradiction.

Case 2. \(b = 0, a \neq 1\). From (3.8), we have
\[F = \frac{G + a - 1}{a}.
Similarly, we also can get a contradiction. Thus \(a = 1\) follows, it implies that \(F = G\).

Case 3. \(b = -1, a \neq -1\). From (3.8), we obtain
\[F = \frac{a}{a + 1 - G}.
Similarly, we can get a contradiction, \(a = -1\) follows. Thus, we get \(F.G = 1\).

From Lemma 2.10, we get \(f = t^g\) for \(t^n = t^{n+1} = 1\), and \(f\) and \(g\) satisfy the algebraic expression \(R(f, g) = 0\), where
\[R(w_1, w_2) = P(w_1)w_1(qz + c) - P(w_2)w_2(qz + c)\]
Thus, we have completed the proofs.
References


