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Elastic effects in superposed fluids

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A non-uniform electric field of suitable gradient can make specific weights of two superposed dielectric fluids identical. If the fluids are Newtonian, this choice of electric field makes the interface resilient to small perturbations, even if the fluid on the top is heavier than the one at bottom. On the other hand, if the fluids are viscoelastic, the interface continues to remain unstable. We point out that although the right choice of electric field succeeds in overcoming the effects of gravity, the fluids' elasticity makes the interface unstable. The same effect can be achieved in the case of paramagnetic or ferro-fluids in presence of a non-uniform magnetic field.
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I. INTRODUCTION

Lord Rayleigh¹ proved that the equilibrium of a stratified liquid is unstable if its density rises with height. Sir Geoffrey Ingram Taylor² demonstrated that this situation is equivalent to a lighter fluid accelerating against a heavier one. In one of our earlier papers³ we showed that a non-uniform electric field alters the specific weight of a fluid. We also showed that it is possible to choose a gradient to equalize the specific weights of superposed fluids. Such a choice makes an arrangement of a heavy dielectric fluid on top of a lighter dielectric fluid resilient to small perturbations of the interface between them. We derived the result for fluids without elasticity. In this paper, I extend that treatment to elastic fluids.

Effects of fluids' elasticity on stability of their flows have been known for a long time. Geisekus⁴ predicted that in the limit of Taylor number Ta going to zero, Taylor-Couette flow of a viscoelastic fluid becomes unstable when the second normal stress difference is positive and large compared with the first. Since the instability was predicted in the regime $Ta \rightarrow 0$, that is, when the inertia was negligible, Geisekus concluded that it was because of the fluid's elasticity. Muller *et al.*⁵ reported an experiment in which they observed a purely elastic instability in a Taylor-Couette flow when the Deborah number of the flow exceeded a critical value. They analyzed the instability within linear regime and found that the critical Deborah number depends on the gap between the rotating cylinders. They also noted that in most polymeric fluids, Geisekus' criterion for instability cannot be fulfilled. Renardy⁶ showed purely elastic instability in two-fluid flows. He showed that, even if the density and viscosity of the superposed fluids match in a plane Couette flow, an elastic stratification is enough to make the interface unstable. Su and Khomami⁷ found similar effects in Poiseuille flow of superposed fluids. They found that the interface between the fluids is unstable if more than half of the channel's height is filled by the fluid with lesser elasticity among the two. In another paper⁸ they show that the instability is a result of a discontinuity in the first normal stress difference at the interface of superposed fluids. Recently, Suman and Kumar⁹ showed that elasticity of liquids manifests as an "effective inertia." They showed that vertical vibrations of a viscoelastic liquid are unstable in the limit of zero inertia. In the same limit, vertical vibrations of a Newtonian fluid were shown to be stable. They concluded that the instability was due to the fluid's elasticity. Elasticity introduced inertia-like terms in their equations, leading to instability of their solution. The idea of

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checking behavior of fluids in the limit of zero inertia was also used in the much older work of Aitken and Wilson.¹⁰ They studied Rayleigh-Taylor instability in Maxwell and Jeffreys fluids. They found that the instability persists even in the limit of zero inertia and concluded that it is caused by elasticity alone. This paper is similar, in the sense that an appropriate electric field gradient eliminates the difference in inertia of the superposed fluids. The instability of the arrangement is therefore attributed to the fluids' elasticity. Saasan and Tyvand¹¹ studied Rayleigh-Taylor instability in a Maxwell fluid and concluded that elasticity of the fluids leads to an increase in the maximum growth rate and corresponding wave numbers, when compared to fluids that are only viscous. Saasan and Hassager¹² extended the treatment to Jeffreys fluid. Piriz *et al.*¹³ studied Rayleigh-Taylor instability in elastic solids. They considered the onset of instability at the interface between two solids and a solid and a fluid. Dimonte¹⁴ studied the effect of elasticity on Rayleigh-Taylor instability and the related Richtmyer-Meshkov instability. He reported that considering fluids to be elastic allowed a better agreement of the theory with observation of geophysical phenomena like formation of salt domes. Joseph *et al.*¹⁵ studied Rayleigh-Taylor instability in the form of breakup of viscoelastic drops. Their theory explained the observations in one of their prior experiments¹⁶ that elasticity of fluids made the drops more susceptible to breakup. In this paper, I will analyze the problem in a more traditional setup, where the viscoelastic fluids are merely superposed on each other (that is, I do not study the breakup of drops). I will show that, contrary to the Newtonian case, matching specific weights of superposed *elastic* fluids does not make an arrangement of a heavy fluid on top of a lighter one stable.

In Sec. II, I review the mathematical analysis of small perturbations in a viscoelastic fluid. I then argue, by using a very simple model, how elasticity of fluids can affect their stability. In Sec. IV, I summarize the effect of a non-uniform electric (magnetic) field on dielectric (paramagnetic) fluids. I then set up the basic equations of the problem and solve them assuming that the perturbations are small enough to ignore quadratic and higher order quantities in the perturbed quantities. Section VI discusses the inferences of the analysis. In Sec. VII, I derive the differential equation for the time evolution of perturbation and apply it to an ideal elastic fluid. Appendix A to the paper describes the stress in fluids due to electric and magnetic fields. I derive an expression for force density from the stress and use it in Appendix B to show that it is possible to set up a non-uniform field to match the specific weights of superposed fluids. Appendix C to this paper carries proofs of theorems mentioned in Sec. V.

II. VISCOELASTIC FLUIDS SUBJECT TO SMALL PERTURBATIONS

Constitutive relations of linear viscoelastic fluids can be written in an integral form as¹⁷

$$\boldsymbol{\tau} = - \int_{-\infty}^t G(t-t') \dot{\boldsymbol{\gamma}}(t') dt', \quad (1)$$

where $G(t-t')$ is the relaxation modulus, $\boldsymbol{\tau}$ is deviatoric part of the total stress tensor $\boldsymbol{\pi}$, and $\dot{\boldsymbol{\gamma}}$ is the rate of strain tensor. I shall use bold Greek symbols to denote second order tensors and bold Roman symbols to denote vectors. I shall also use a convention, popular in Physics, of writing the total stress tensor as

$$\boldsymbol{\pi} = p\boldsymbol{\delta} + \boldsymbol{\tau}, \quad (2)$$

where p is the pressure and $\boldsymbol{\delta}$ is a unit tensor or rank 2. The law of conservation of momentum for viscoelastic fluids is given by the Cauchy equation,

$$\rho \dot{\mathbf{u}} = -\nabla \cdot \boldsymbol{\pi} + \mathbf{f}. \quad (3)$$

In this equation, ρ is the density of the fluid, $\mathbf{u}$ is the velocity of a fluid parcel, $\dot{\mathbf{u}}$ is its convective derivative, and $\mathbf{f}$ is the density of a body force. Using Eqs. (1) and (2) in Eq. (3) and assuming that the only body force is that due to gravity, we have

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \int_{-\infty}^t G(t-t') \nabla \cdot \dot{\boldsymbol{\gamma}}(t') dt' + \rho \mathbf{g}. \quad (4)$$

I will now illustrate that the entire machinery, developed for analyzing Rayleigh-Taylor instability in Newtonian fluids,¹⁸ can be reused for linear viscoelastic fluids by replacing the constant viscosity μ of Newtonian fluids with a complex viscosity η^* characterizing a viscoelastic fluid. The illustration is an adaptation of a result about behavior of viscoelastic fluids under small oscillations, proved in the book of Bird *et al.*¹⁹ Let the velocity field be perturbed so that it is expressed as $\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)}$, the superscript “(0)” indicating the base flow and the superscript “(1)” indicating perturbation. Let the pressure and density be written in a similar fashion. If the perturbations are small, quadratic and higher order terms in perturbed quantities can be ignored to give,

$$\rho^{(0)} \frac{\partial \mathbf{u}^{(1)}}{\partial t} = -\nabla p^{(1)} + \int_{-\infty}^t G(t-t') \nabla \cdot \dot{\boldsymbol{\gamma}}^{(1)} dt' + \rho^{(1)} \mathbf{g}, \quad (5)$$

where $\dot{\boldsymbol{\gamma}}^{(1)}$ is the rate of strain tensor due to perturbation alone and I have used the fact that for a fluid initially at rest $\mathbf{u}^{(0)} = 0$. If the fluid is incompressible,

$$\nabla \cdot \dot{\boldsymbol{\gamma}}^{(1)} = \Delta \mathbf{u}^{(1)},$$

where Δ is the Laplacian operator. Therefore,

$$\rho^{(0)} \frac{\partial \mathbf{u}^{(1)}}{\partial t} = -\nabla p^{(1)} + \int_{-\infty}^t G(t-t') \Delta \mathbf{u}^{(1)}(t') dt' + \rho^{(1)} \mathbf{g}. \quad (6)$$

Assume a normal mode expansion of the form $\mathbf{u}^{(1)} = \mathbf{A} \exp(nt + i\mathbf{k} \cdot \mathbf{x})$, where n is complex and only the real part of the expression is physically significant. The perturbation in pressure field is similarly written as $p^{(1)} = B \exp(nt + i\mathbf{k} \cdot \mathbf{x})$. Substituting it in the above equation we get

$$n\rho^{(0)} \mathbf{u}^{(1)} = -\mathbf{k} p^{(1)} - k^2 \int_{-\infty}^t G(t-t') \mathbf{u}^{(1)}(t') dt' + \rho^{(1)} \mathbf{g}. \quad (7)$$

Equation (7) can be written in a simpler form as

$$n\rho^{(0)} \mathbf{u}^{(1)} = -\mathbf{k} p^{(1)} - k^2 \eta^* \mathbf{u}^{(1)} + \rho^{(1)} \mathbf{g}, \quad (8)$$

where the complex viscosity, η^* (the superscript * denotes the complex conjugate) is defined as¹⁹

$$\eta^* = \int_0^\infty G(s) \exp(-ns) ds. \quad (9)$$

I can include the effect of surface tension in Eq. (6) by adding a term $-(\gamma \nabla \cdot \mathbf{n}^{(1)}) \mathbf{e}_z \delta(z - z^{(0)})$, where γ is the interfacial tension, $\mathbf{n}^{(1)}$ is the normal to the perturbed interface, $\delta(\cdot)$ is the Dirac delta function, and the unperturbed interface is at $z = z^{(0)}$, to get

$$\rho^{(0)} \frac{\partial \mathbf{u}^{(1)}}{\partial t} = -\nabla p^{(1)} + \int_{-\infty}^t G(t-t') \Delta \mathbf{u}^{(1)}(t') dt' + \rho^{(1)} \mathbf{g} - (\gamma \nabla \cdot \mathbf{n}^{(1)}) \mathbf{e}_z \delta(z - z^{(0)}). \quad (10)$$

III. A SIMPLE MODEL OF ELASTIC INSTABILITY

I will demonstrate the influence of elasticity on stability of fluids through a simple model. Consider a stratified fluid whose density depends on the z coordinate. Let the gravitational acceleration also be along the z axis. Let a fluid parcel at a point z be exchanged with the one at $z + h$. If the fluid is incompressible, the volume of the parcel does not change as it gets to the position $z + h$. Since it displaces a fluid of density $\rho(z + h)$, it experiences a buoyancy force $(\rho(z) - \rho(z + h))g$. If the fluid is elastic, the fluid parcel will also experience an additional force $-k_1 h$, where k_1 is a parameter describing elasticity of the fluid. The fluid's viscosity will contribute a drag force of the form $-k_2 dh/dt$, where k_2 is a damping constant. Newton's second law, applied to the fluid

parcel gives

$$\rho \frac{d^2 h}{dt^2} = -(\rho(z+h) - \rho(z))g - k_1 h - k_2 \frac{dh}{dt}. \quad (11)$$

For small h , up to which the linear form of elastic force holds good, we can ignore higher than linear terms in the Taylor expansion of $\rho(z+h)$ to get

$$\rho \frac{d^2 h}{dt^2} = -\left(\frac{d(\rho g)}{dz} + k_1\right)h - k_2 \frac{dh}{dt}. \quad (12)$$

The quantity ρg on the right hand side is the specific weight.

If the fluid were not elastic and an electric (or magnetic) field was chosen to erase the variation in specific weight then viscous drag will cause a displaced parcel to slow down at an exponential rate. Presence of the elastic force introduces an oscillatory behavior to the fluid parcel's motion. The oscillatory component makes the motion susceptible to resonance with modes of perturbation matching with the oscillation's natural frequency. Thus, an arrangement that was stable in absence of elasticity, is now unstable.

IV. FLUID IN ELECTRIC OR MAGNETIC FIELDS

The effect of a non-uniform, static, electric, or magnetic field, along or against the direction of gravity, on a fluid can be expressed in terms of an effective acceleration due to gravity. The fields change the specific weight of the fluids. Our previous papers^{3,20} had derived an expression for the effect. However, there was an error in one of the formulas. I therefore quote the corrected formulas in this section and derive them once again in Appendix B to this paper.

If a fluid is exposed to a non-uniform electric field $\mathbf{E} = E(z)\mathbf{e}_z$, $\mathbf{e}_z$ denoting a unit vector along the z axis, then it experiences an effective acceleration due to gravity given by

$$\check{g}_e = g + 2K_e^2 \frac{E(0^-)}{\rho} \frac{\partial E}{\partial z}, \quad (13)$$

where the constants K_e depends only on the molecular and bulk properties of the fluids. It is expressed as

$$K_e^2 = \epsilon_0 \rho \left(\frac{\partial \kappa_e}{\partial \rho} \right)_T - \epsilon_0 \frac{\kappa_e}{2}, \quad (14)$$

where ρ is the fluid's density, κ_e is the relative permittivity, ϵ_0 is the permittivity of free space, and T is the absolute temperature. The fluid's permittivity is $\epsilon = \kappa_e \epsilon_0$.

If a fluid is exposed to a non-uniform magnetizing field $\mathbf{H} = H(z)\mathbf{e}_z$, then it experiences an effective acceleration due to gravity given by

$$\check{g}_m = g + 2K_m^2 \frac{H(0^-)}{\rho} \frac{\partial H}{\partial z}, \quad (15)$$

where the constants K_m depends only on the molecular and bulk properties of the fluids. It is expressed as

$$K_m^2 = \mu_0 \rho \left(\frac{\partial \kappa_m}{\partial \rho} \right)_T - \mu_0 \frac{\kappa_m}{2}, \quad (16)$$

where ρ is the fluid's density, κ_m is the relative permeability, μ_0 is the permeability of free space, and T is the absolute temperature. The fluid's permeability is $\mu = \kappa_m \mu_0$.

Since the pair of equations describing the effect of electric fields is very similar to the pair describing the effect of magnetic fields, we can develop a theory for both simultaneously, using an effective acceleration due to gravity $\check{g}$. It will be equal to $\check{g}_e$ in the case of an electric field and $\check{g}_m$ in the case of a magnetic field.

V. LINEAR ANALYSIS

Let us consider perturbations in a stratified fluid at rest whose density varies in the z direction alone. We can use the results of this analysis to the case of superposed fluids, each with a uniform density, by assuming that the function $\rho(z)$ is discontinuous at their interface $z = z^{(0)}$ and by adding an interfacial tension term. Let the components of perturbed velocity $\mathbf{u}^{(1)}$ be $(u^{(1)}, v^{(1)}, w^{(1)})$. We assume the fluid to be incompressible. Therefore, $\nabla \cdot \mathbf{u} = 0$. Since $\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)}$, and $\mathbf{u}^{(0)} = 0$, the condition of incompressibility gives

$$\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} + \frac{\partial w^{(1)}}{\partial z} = 0. \quad (17)$$

The equation of continuity, after assuming $u^{(0)} = 0$ and using Eq. (17), is

$$\frac{\partial \rho^{(1)}}{\partial t} + w^{(1)} \frac{\partial \rho^{(0)}}{\partial z} = 0. \quad (18)$$

Assuming a normal mode expansion of the form $\exp(nt + i\mathbf{k} \cdot \mathbf{r})$, where n is complex and only the real part of the expression is physically significant, and writing Eq. (6) in component form

$$\rho^{(0)} n u^{(1)} = -ik_x p^{(1)} + \int_{-\infty}^t G(t-t')(D^2 - k^2)u^{(1)}(t')dt', \quad (19)$$

$$\rho^{(0)} n v^{(1)} = -ik_y p^{(1)} + \int_{-\infty}^t G(t-t')(D^2 - k^2)v^{(1)}(t')dt', \quad (20)$$

$$\rho^{(0)} n w^{(1)} = -Dp^{(1)} + \int_{-\infty}^t G(t-t')(D^2 - k^2)w^{(1)}(t')dt' - \rho^{(1)} \check{g}. \quad (21)$$

Similarly, Eq. (17) becomes

$$ik_x u^{(1)} + ik_y v^{(1)} + Dw^{(1)} = 0, \quad (22)$$

where $D \equiv d/dz$ and Eq. (18) becomes

$$n\rho^{(1)} + w^{(1)}D\rho^{(0)} = 0. \quad (23)$$

Multiplying Eq. (19) by $-ik_x$, Eq. (20) by $-ik_y$, adding them and using (22) in the result, we get

$$\rho^{(0)} n Dw^{(1)} = -k^2 p^{(1)} + \int_{-\infty}^t G(t-t')(D^2 - k^2)Dw^{(1)}dt', \quad (24)$$

where $k^2 = (k_x^2 + k_y^2)$. Theorem 1 (proved in Appendix C) allows us to write the last term of Eq. (24) in terms of complex viscosity as

$$\rho^{(0)} n Dw^{(1)} = -k^2 p^{(1)} + \eta^*(D^2 - k^2)Dw^{(1)}. \quad (25)$$

Using Eq. (23) we can express $\rho^{(1)}$ as $-w^{(1)}D\rho^{(0)}/n$ in (21). If we also write the result in terms of complex viscosity, we get

$$\rho^{(0)} n w^{(1)} = -Dp^{(1)} + \eta^*(D^2 - k^2)w^{(1)} + \frac{\check{g}w^{(1)}D\rho^{(0)}}{n}. \quad (26)$$

I can include the case of superposed fluids by adding a term corresponding to interfacial tension to the above equation. If the interface at $z = z^{(0)}$ is perturbed by a quantity ξ then the normal to the perturbed interface is²¹

$$\mathbf{n}^{(1)} = -\frac{\nabla \xi}{|\nabla \xi|} = -\frac{\xi_{,x}\mathbf{e}_x + \xi_{,y}\mathbf{e}_y - \mathbf{e}_z}{\sqrt{\xi_{,x}^2 + \xi_{,y}^2 + 1}}, \quad (27)$$

where the subscript notation for partial derivatives is used. That is, ξ_x is the partial derivative of ξ with respect to x . I use Eq. (10) instead of Eq. (6) to derive Eqs. (19)–(21). Continuing in same

manner, I find that Eq. (26) is modified to

$$\rho^{(0)}nw^{(1)} = -Dp^{(1)} + \eta^*(D^2 - k^2)w^{(1)} + \frac{\check{g}w^{(1)}D\rho^{(0)}}{n} - k^2\gamma z^{(1)}\delta(z - z^{(0)}). \quad (28)$$

Let the fluids be confined between two rigid boundaries $z = 0$ and $z = d$. Since the tangential and normal velocities vanish at solid boundaries, we have $w^{(1)} = 0$ and $Dw^{(1)} = 0$ at both, $z = 0$ and $z = d$. Equations (25) and (28) are coupled eigenvalue relations for the functions $w^{(1)}$ and $p^{(1)}$ with eigenvalue n . Let n_i and n_j be two eigenvalues with the corresponding eigenfunctions $(w_i^{(1)}, p_i^{(1)})$ and $(w_j^{(1)}, p_j^{(1)})$. For the eigenvalue n_i , Eq. (28) is

$$Dp_i^{(1)} = -n_i\rho^{(0)}w_i^{(1)} + \eta^*(D^2 - k^2)w_i^{(1)} + \frac{\check{g}w_i^{(1)}D\rho^{(0)}}{n_i} - k^2\gamma z^{(1)}\delta(z - z^{(0)}). \quad (29)$$

Since

$$w_i^{(1)} = \frac{dz^{(1)}}{dt}, \quad (30)$$

the normal mode expansion of $z^{(1)}$ gives $w_i^{(1)} = z^{(1)}n_i$. Therefore,

$$Dp_i^{(1)} = -n_i\rho^{(0)}w_i^{(1)} + \eta^*(D^2 - k^2)w_i^{(1)} + \frac{\check{g}w_i^{(1)}D\rho^{(0)}}{n_i} - k^2\gamma \frac{w_i^{(1)}}{n_i}\delta(z - z^{(0)}). \quad (31)$$

Multiply by $w_j^{(1)}$ and integrate between $z = 0$ and $z = d$, to get

$$\begin{aligned} \int_0^d w_j^{(1)} Dp_i^{(1)} dz &= -n_i \int_0^d \rho^{(0)} w_i^{(1)} w_j^{(1)} dz + \eta^* \int_0^d [(D^2 - k^2)w_i^{(1)}] w_j^{(1)} dz + \\ &\quad \frac{\check{g}}{n_i} \int_0^d w_j^{(1)} w_i^{(1)} D\rho^{(0)} dz - \frac{k^2\gamma}{n_i} \int_0^d w_j^{(1)} w_i^{(1)} \delta(z - z^{(0)}) dz. \end{aligned} \quad (32)$$

The boundary conditions on $w_j^{(1)}$ allow us to write the left hand side as

$$- \int_0^d p_i^{(1)} Dw_j^{(1)} dz,$$

which, after substituting for $p_i^{(1)}$ from Eq. (25) becomes

$$- \int_0^d \left\{ \left[\frac{-n_i\rho^{(0)}}{k^2} + \frac{\eta^*}{k^2}(D^2 - k^2) \right] Dw_i^{(1)} \right\} Dw_j^{(1)} dz = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_0^d \left[\frac{n_i\rho^{(0)}}{k^2} + \eta^* \right] Dw_i^{(1)} Dw_j^{(1)} dz, \\ I_2 &= - \int_0^d \left\{ \frac{\eta^*}{k^2} D^2(Dw_i^{(1)}) \right\} Dw_j^{(1)} dz. \end{aligned} \quad (33)$$

The form of I_2 can be simplified after integrating by parts to

$$I_2 = \frac{\eta^*}{k^2} \int_0^d (D^2 w_i^{(1)})(D^2 w_j^{(1)}) dz.$$

Equation (32), therefore becomes

$$\begin{aligned} \int_0^d \left[\frac{n_i \rho^{(0)}}{k^2} + \eta^* \right] Dw_i^{(1)} Dw_j^{(1)} dz + \frac{\eta^*}{k^2} \int_0^d (D^2 w_i^{(1)})(D^2 w_j^{(1)}) dz = \\ -n_i \int_0^d \rho^{(0)} w_i^{(1)} w_j^{(1)} dz + \eta^* \int_0^d [(D^2 - k^2) w_i^{(1)}] w_j^{(1)} dz \\ + \frac{\check{\gamma}}{n_i} \int_0^d w_j^{(1)} w_i^{(1)} D\rho^{(0)} dz - \frac{k^2 \gamma}{n_i} \int_0^d w_j^{(1)} w_i^{(1)} \delta(z - z^{(0)}) dz \end{aligned} \quad (34)$$

or

$$\begin{aligned} -n_i \int_0^d \left\{ \rho^{(0)} w_i^{(1)} w_j^{(1)} + \frac{\rho^{(0)}}{k^2} Dw_i^{(1)} Dw_j^{(1)} \right\} dz + \frac{\check{\gamma}}{n_i} \int_0^d w_j^{(1)} w_i^{(1)} D\rho^{(0)} dz \\ - \frac{k^2 \gamma}{n_i} \int_0^d w_j^{(1)} w_i^{(1)} \delta(z - z^{(0)}) dz = \\ \eta^* \int_0^d \left\{ Dw_i^{(1)} Dw_j^{(1)} + \frac{x}{k^2} (D^2 w_i^{(1)})(D^2 w_j^{(1)}) - ((D^2 - k^2) w_i^{(1)}) w_j^{(1)} \right\} dz. \end{aligned} \quad (35)$$

Simplifying the last term on the right hand side and integrating the result by parts we have

$$\begin{aligned} -n_i \int_0^d \left\{ \rho^{(0)} w_i^{(1)} w_j^{(1)} + \frac{\rho^{(0)}}{k^2} Dw_i^{(1)} Dw_j^{(1)} \right\} dz + \frac{\check{\gamma}}{n_i} \int_0^d w_j^{(1)} w_i^{(1)} D\rho^{(0)} dz \\ - \frac{k^2 \gamma}{n_i} \int_0^d w_j^{(1)} w_i^{(1)} \delta(z - z^{(0)}) dz = \\ \eta^* \int_0^d \left\{ k^2 w_i^{(1)} w_j^{(1)} + 2 Dw_i^{(1)} Dw_j^{(1)} + \frac{x}{k^2} (D^2 w_i^{(1)})(D^2 w_j^{(1)}) \right\} dz. \end{aligned} \quad (36)$$

Equation (36) leads us to the following two theorems (proved in Appendix C):

1. For an eigenvalue n_i of Eq. (36) we have

$$(n_i^2 - (n_i^*)^2) I_3 = (n_i^* \eta - n_i \eta^*) I_4, \quad (37)$$

where

$$I_3 = \int_0^d \left\{ \rho^{(0)} |w_i^{(1)}|^2 + \frac{\rho^{(0)}}{k^2} |Dw_i^{(1)}|^2 \right\} dz > 0, \quad (38)$$

$$I_4 = \int_0^d \left\{ k^2 |w_i^{(1)}|^2 + 2 |Dw_i^{(1)}|^2 + \frac{|D^2 w_i^{(1)}|^2}{k^2} \right\} dz > 0. \quad (39)$$

2. For an eigenvalue n_i of Eq. (36) we have

$$(n_i^* - n_i) I_5 + \left(\frac{\check{\gamma}}{n_i} - \frac{\check{\gamma}}{n_i^*} \right) I_6 - \left(\frac{k^2 \gamma}{n_i} - \frac{k^2 \gamma}{n_i^*} \right) I_7 = 0, \quad (40)$$

where

$$I_5 = \int_0^d \left\{ \rho^{(0)} |w_i^{(1)}|^2 + \frac{\rho^{(0)}}{k^2} |Dw_i^{(1)}|^2 \right\} dz > 0, \quad (41)$$

$$I_6 = \int_0^d |w_i^{(1)}|^2 D\rho^{(0)} dz, \quad (42)$$

$$I_7 = \int_0^d |w_i^{(1)}|^2 \delta(z - z^{(0)}) dz > 0. \quad (43)$$

The integral I_6 depends on whether density of the fluids increases or decreases with height. $I_6 > 0$ when the fluid is heavier at the top.

VI. DISCUSSION OF RESULTS

The main results of the analysis of Sec. V are Eqs. (37) and (40). Writing them together once again

$$(n_i^2 - (n_i^*)^2)I_3 = (n_i^*\eta - n_i\eta^*)I_4, \quad (44)$$

$$(n_i^* - n_i) \left(I_5 + \frac{\check{g}I_6}{|n_i|^2} - \frac{k^2\gamma I_7}{|n_i|^2} \right) = 0. \quad (45)$$

I will first check if they reduce to known facts in simpler cases. For example,

1. If the fluids were Newtonian instead of viscoelastic, $\eta = \eta^*$ and Eq. (44) will imply

$$\text{Re}(n_i) = -\frac{I_4}{2I_3}\eta < 0. \quad (46)$$

Equation (44) was derived under the assumption that n_i was complex. Therefore, Eq. (46) tells us that oscillatory modes, if they exist, are stable. It tallies with Chandrasekhar's¹⁸ inference from his Eq. (86) of section (93).

2. Suppose the fluids are Newtonian and $I_6 > 0$, that is the density of fluid rises with z . Further, suppose that there are no electric or magnetic fields. That is $\check{g} = g$. Equation (45) implies that one of the following equations is true:

$$(n_i^* - n_i) = 0, \quad (47)$$

$$\left(I_5 + \frac{gI_6}{|n_i|^2} - \frac{k^2\gamma I_7}{|n_i|^2} \right) = 0. \quad (48)$$

If there were no surface tension, as in the case of a stratified fluid, Eq. (48) cannot be true. Therefore, the only way to satisfy Eq. (45) is to insist on truth of Eq. (47). Which means that if $I_6 > 0$, there are no oscillatory modes. These equations, on their own, are not enough to tell whether the non-oscillatory modes are stable or not. Chandrasekhar's treatise¹⁸ shows that they are unstable.

3. In the case of superposed Newtonian fluids such that the heavier fluid is at the top, $I_6 > 0$ and $\gamma \neq 0$. If we choose the electric (or magnetic) field gradient such that $\check{g} = 0$ then Eq. (45) can be true even without insisting that n_i be real. However, even in this case, because η is real, Eq. (46) is true and oscillatory modes stay stable. This tallies with our earlier result³ that it is possible to choose an electric field gradient such that an arrangement of a heavy Newtonian fluid on top of a lighter one can be made stable.

Let us now apply Eqs. (44) and (45) to viscoelastic fluids. Now η is complex. Therefore, the only information we can get from Eq. (44) is by writing $n_i = a + bi$ to get

$$b = \frac{a\text{Im}(\eta)I_4}{2aI_3 + \text{Re}(\eta)I_4}. \quad (49)$$

1. If we assume that the upper fluid is lighter then $I_6 < 0$. Equation (45) can be satisfied if any one or both of Eqs. (47) and (48) are true. Let us, for the moment, assume that Eq. (48) is true. Therefore,

$$|n_i|^2 = (a^2 + b^2) = \left(\frac{k^2\gamma I_7 - gI_6}{I_5} \right) = \left(\frac{k^2\gamma I_7 + g|I_6|}{I_5} \right). \quad (50)$$

We now use the form of b derived in Eq. (49) to get a quartic equation in a ,

$$a^4 I_3^2 + a^3 I_3 I_4 \text{Re}(\eta) + \left(|\eta|^2 \frac{I_4^2}{4} - |n_i|^2 I_3^2 \right) a^2 - |n_i|^2 I_3 I_4 \text{Re}(\eta) a - \frac{|n_i|^2 \text{Re}^2(\eta) I_4^2}{4} = 0. \quad (51)$$

Since $|n_i|^2 > 0$, by Descartes' rule of signs, mentioned in Appendix C to this paper, Eq. (51) has at least one positive root. Therefore, oscillatory modes could be unstable. However, on physical grounds, we know that perturbations in a fluid stratified such that it is heavier at the bottom will always fade. We are therefore left with no choice but to assume that Eq. (47) is true, which means that n_i is real. Thus, there are no oscillatory modes even if the lower fluid is heavier than the upper one.

2. I will now examine the main result of this paper. Suppose the fluids are viscoelastic and $I_6 > 0$, that is the density of fluid rises with z . Further, let the stratification be due to superposition of uniform fluids of differing densities. Therefore, γ cannot be zero. The stratification, left to itself, is unstable and the instability is due to a heavy fluid on top of a lighter one. It is possible to erase this difference by choosing a field gradient such that $\check{g} = 0$ at the interface. In that case, Eq. (45) can be satisfied without forcing n_i to be real. The real part of n_i then satisfies the quartic equation (51) that has at least one positive root. The corresponding perturbation modes are unstable. This instability is solely due to elastic nature of the fluids.

VII. INSTABILITY IN AN IDEAL ELASTIC FLUID

In Sec. III, I presented a very simple model of how elasticity can cause instability in a stratified fluid. In this section, I will derive a differential equation of evolution of $w^{(1)}$ and show how elasticity makes certain arrangement of fluids unstable, which would otherwise have been stable. Using Eq. (25) in Eq. (28) I get

$$D \left\{ \rho^{(0)} D w^{(1)} - \frac{\eta^*}{n} (D^2 - k^2) D w^{(1)} \right\} = k^2 \left\{ \rho^{(0)} w^{(1)} - \frac{\eta^*}{n} (D^2 - k^2) w^{(1)} \right\} + k^2 \left\{ -\frac{\check{g} w^{(1)} D \rho^{(0)}}{n^2} + \frac{k^2}{n^2} \gamma w^{(1)} \delta(z - z^{(0)}) \right\}. \quad (52)$$

This equation is analogous to Eq. (41) of Chandrasekhar's¹⁸ section 91, except that in this case the viscosity is a complex constant. I will solve it with the following conditions at the interface $z = z^{(0)}$ between the fluids,

1. $w^{(1)}$ is continuous;
2. $D w^{(1)}$ is continuous; and
3. $\eta^* (D^2 + k^2) w^{(1)}$ is continuous.

To examine the effect of elasticity on the instability, let me consider the fluids to be inviscid but elastic, that is, $\eta^* = -i\eta''$. In the regions of constant $\rho^{(0)}$ and η^* , Eq. (52) simplifies to

$$\left(1 - \frac{\nu^*}{n} (D^2 - k^2) \right) (D^2 - k^2) w^{(1)} = 0, \quad (53)$$

where the complex kinematic viscosity is

$$\nu^* = \nu' - i\nu'' = \frac{\eta^*}{\rho^{(0)}}. \quad (54)$$

In the case of an ideal elastic fluid, Eq. (52) becomes

$$\left(1 - \frac{(-i\nu'')}{n} (D^2 - k^2) \right) (D^2 - k^2) w^{(1)} = 0. \quad (55)$$

Following Chandrashekar,¹⁸ I assume that the two fluids have same complex kinematic viscosity to get the dispersion relation,

$$y^4 + 4\beta y^3 + 2(1 - 6\beta)y^2 - 4(1 - 3\beta)y + (1 - 4\beta) + Q(\alpha_1 - \alpha_2) + Q^{1/3}S = 0, \quad (56)$$

where

$$y = \sqrt{x + 1}, \quad (57)$$

$$x = \frac{n}{k^2(-iv'')}, \quad (58)$$

$$Q = \frac{\check{g}}{k^3(-iv'')^2}, \quad (59)$$

$$S = \frac{\gamma}{(\rho_1^{(0)} + \rho_2^{(0)})(\check{g}v''^4)^{1/3}}, \quad (60)$$

$$\beta = \alpha_1\alpha_2, \quad (61)$$

$$\alpha_1 = \frac{\rho_1^{(0)}}{(\rho_1^{(0)} + \rho_2^{(0)})}, \quad (62)$$

$$\alpha_2 = \frac{\rho_2^{(0)}}{(\rho_1^{(0)} + \rho_2^{(0)})}. \quad (63)$$

The reader may refer to section 94 of Chandrasekhar's treatise¹⁸ for the steps leading to the dispersion relation from the differential equation and its boundary conditions. If we choose the electric field so that $\check{g} = 0$, the dispersion relation simplifies to

$$(y - 1)(y^3 + (4\beta + 1)y^2 + (3 - 8\beta)y + (4\beta - 1)) = 0. \quad (64)$$

The nature of the instability depends on the roots of the cubic equation $(y^3 + (4\beta + 1)y^2 + (3 - 8\beta)y + (4\beta - 1)) = 0$. From Eqs. (61), (62), and (64), it is clear that $0 < \beta \leq 0.25$. For this range of β , the discriminant of the cubic equation $(-1024\beta^3 - 2048\beta^2 + 1280\beta - 276)$ is negative. Therefore, the cubic equation has one real root and two complex roots. The complex roots are conjugates of each other.

The interface between the fluids is unstable if $\text{Re}(n) > 0$. Using Eqs. (57) and (58), this condition is equivalent to $\text{Re}(y)\text{Im}(y) > 0$. When there are two complex roots, conjugates of each other, it is easy to verify that for one of them $\text{Re}(y)\text{Im}(y) > 0$. Alternatively, using Routh-Hurwitz theorem,²² one can prove that $\text{Re}(y) < 0$. Therefore, if $a + |b|i$ and $a - |b|i$ are two roots, then for the second root, since $a = \text{Re}(y) < 0$, $\text{Re}(y)\text{Im}(y) > 0$.

If I were to repeat this analysis for fluids without elasticity, then $\text{Re}(n) > 0$ if $(\text{Re}^2(y) - \text{Im}^2(y) - 1) > 0$. There is no analytical criterion to decide the sign of $(\text{Re}^2(y) - \text{Im}^2(y) - 1)$. However, a numerical search resulted in no such y , indicating that the arrangement of fluids is stable.

APPENDIX A: ELECTRIC AND MAGNETIC STRESS IN FLUIDS

1. Volume forces in dielectric fluids

Stress inside a fluid dielectric in presence of an electric field $\mathbf{E}$, in SI units, is given by²³

$$\boldsymbol{\pi}_e = p\boldsymbol{\delta} + \left(\frac{\epsilon}{2}E^2 - \rho \frac{E^2}{2} \frac{\partial \epsilon}{\partial \rho} \right) \boldsymbol{\delta} - \epsilon \mathbf{E} \mathbf{E}. \quad (\text{A1})$$

It reduces to Maxwell stress tensor if the electric field is in vacuum. In that case, $\epsilon = \epsilon_0$, the constant permittivity of free space, and there is no hydrostatic pressure. Equation (A1) then becomes

$$\boldsymbol{\pi}_e = \left(\frac{\epsilon_0}{2}E^2 \right) \boldsymbol{\delta} - \epsilon_0 \mathbf{E} \mathbf{E}. \quad (\text{A2})$$

Maxwell stress tensor arises in the equation of conservation of linear momentum of a system of charges and fields. A closer look at the development of momentum conservation law tells us that the tensor is applicable only to vacuum macroscopic and microscopic fields.²⁴ Using it in the case of dielectric fluid in an electric field is tantamount to ignoring its material properties.

The stress tensor of Eq. (A1) can be used directly if we are interested in surface boundary conditions or surface effects. For example, in the case of a thin polymer film sandwiched between parallel electrode plates, but with an air gap between the film and the top electrode, the continuity of stress gives

$$\boldsymbol{\pi}_e \cdot \mathbf{n} = \boldsymbol{\pi}'_e \cdot \mathbf{n}, \quad (\text{A3})$$

where $\boldsymbol{\pi}_e$ is the stress tensor inside the film, $\boldsymbol{\pi}'_e$ is that in the air gap, and $\mathbf{n}$ is a unit normal to the film's surface. The air gap can be treated as a vacuum. In that case, if E'_n and E'_t are the normal and tangential components (with respect to the film's plane) of the electric field, then

$$\boldsymbol{\pi}'_e \cdot \mathbf{e}_3 = \frac{\epsilon_0}{2}(E_n'^2 + E_t'^2)\mathbf{e}_3, \quad (\text{A4})$$

while

$$\boldsymbol{\pi}_e \cdot \mathbf{e}_3 = \left(p_0 - \rho \frac{E^2}{2} \frac{\partial \epsilon}{\partial \rho} \right) \mathbf{e}_3 + \frac{\epsilon_0 \kappa}{2} (E_n^2 - E_t^2) \mathbf{e}_3. \quad (\text{A5})$$

If there is no tangential field, $E_t = E'_t = 0$. The boundary condition of continuity of normal component of electric displacement gives $\kappa E_n = E'_n$. This gives the pressure at the upper surface of the film as

$$p_0 = \rho \frac{E^2}{2} \frac{\partial \epsilon}{\partial \rho} - \frac{\epsilon_0 E^2}{2} \kappa (\kappa - 1). \quad (\text{A6})$$

This agrees with the expression used in the paper by Schäffer *et al.*,²⁵ although they ignore the first term on the right hand side. Landau and Lifshitz,²⁶ however, include both the terms.

Many times, one is interested in knowing the effect of electric field on a volume element of the fluid. In such situations, one needs the volume force density. It can be derived from the stress as

$$\mathbf{f}_e = -\nabla \cdot \boldsymbol{\pi}_e. \quad (\text{A7})$$

Using Eq. (A1) and the fact that for a static electric field $\nabla \times \mathbf{E} = 0$, we get

$$\mathbf{f}_e = -\nabla p + \rho_f \mathbf{E} - \frac{\epsilon_0 E^2}{2} \nabla \kappa_e + \frac{\epsilon_0}{2} \nabla \left(E^2 \rho \frac{\partial \kappa_e}{\partial \rho} \right), \quad (\text{A8})$$

where κ_e is the relative permittivity, that is, $\epsilon = \kappa_e \epsilon_0$ and ρ_f is the density of free charges. While deriving Eq. (A8) I used the Gauss law $\nabla \cdot \mathbf{D} = \rho_f$, where the electric displacement $\mathbf{D} = \epsilon \mathbf{E}$. An ideal dielectric fluid does not have free charges, therefore, $\rho_f = 0$ and we get the expression for density of body force as

$$\mathbf{f}_e = -\nabla p - \frac{\epsilon_0 E^2}{2} \nabla \kappa_e + \frac{\epsilon_0}{2} \nabla \left(E^2 \rho \frac{\partial \kappa_e}{\partial \rho} \right). \quad (\text{A9})$$

In this expression, the second term on the right hand side is non-zero only if there is a permittivity gradient, which happens only in the case of superposed fluids, stratified fluids or if there is a temperature gradient in the fluid. The third term on the right hand side is called the electrostriction force. It can be simplified using the Clausius-Mossotti relation,²⁷

$$\frac{\kappa_e - 1}{\kappa_e + 2} = \frac{N_a \rho \alpha}{3M}, \quad (\text{A10})$$

where N_a is the Avogadro constant, α is the molecular polarizability, and M is the molecular weight. Equation (A10) immediately gives

$$\rho \frac{\partial \kappa_e}{\partial \rho} = \frac{\kappa_e^2 + \kappa_e - 2}{3}. \quad (\text{A11})$$

Thus, the electrostriction force is non-zero only if the electric field and/or the electric permittivity are non-uniform.

2. Volume forces in paramagnetic fluids

Stress inside a paramagnetic fluid in presence of a static magnetization field $\mathbf{H}$ is likewise given by

$$\boldsymbol{\pi}_m = p\boldsymbol{\delta} + \left(\frac{\mu}{2} H^2 - \rho \frac{H^2}{2} \frac{\partial \mu}{\partial \rho} \right) \boldsymbol{\delta} - \mu \mathbf{H} \mathbf{H}. \quad (\text{A12})$$

It reduces to Maxwell stress tensor only if there is no matter in the region of the magnetic field. The volume force $\mathbf{f}_m$ due to the stress tensor is $\mathbf{f}_m = -\nabla \cdot \boldsymbol{\pi}_m$. Using Eq. (A12), the fact that for a static magnetization field $\nabla \times \mathbf{H} = \mathbf{J}$, and the fact that $\nabla \cdot \mathbf{B} = \nabla \cdot (\mu \mathbf{H}) = 0$, we get

$$\mathbf{f}_m = -\nabla p + \mathbf{J} \times \mathbf{B} - \frac{\mu_0 H^2}{2} \nabla \kappa_m + \frac{\mu_0}{2} \nabla \left(H^2 \rho \frac{\partial \kappa_m}{\partial \rho} \right), \quad (\text{A13})$$

where κ_m is the relative permittivity, that is, $\mu = \kappa_m \mu_0$ and $\mathbf{J}$ is the current density. If there is no free current in the fluids, the expression for density of body force is

$$\mathbf{f}_m = -\nabla p - \frac{\mu_0 H^2}{2} \nabla \kappa_m + \frac{\mu_0}{2} \nabla \left(H^2 \rho \frac{\partial \kappa_m}{\partial \rho} \right). \quad (\text{A14})$$

The Clausius-Mossotti relation is valid for relative magnetic permeability as well.

APPENDIX B: EFFECT OF ELECTRIC AND MAGNETIC FIELDS ON SPECIFIC WEIGHT

I demonstrate the effect of a gradient of an applied electric or magnetic field on the fluid's specific weight. To do so, I employ the example of linear analysis of instability of an interface between perfect fluids. Developing the framework for perfect fluids allows us to focus on the effect of fields without getting obscured by the peculiarities of non-Newtonian constitutive relations. Further, the analysis is derived for the general case of Kelvin-Helmholtz instability. I shall derive the conditions for Rayleigh-Taylor instability as a special case. I shall also demonstrate that the mathematical structure of the electric field problem is similar to that of the magnetic field and that a single framework that analyzes both can be developed. Although this analysis was presented in one of our prior papers,²⁰ there was an error in it. I present the analysis again after correcting the error.

Let us consider the stability of the interface between perfect fluids in presence of electric and magnetic fields. Let ρ_1 and ρ_2 be densities of lower and upper fluids, respectively. Let both of them have a uniform base velocity $\mathbf{u}_1^{(0)} = U_1 \mathbf{e}_x$ and $\mathbf{u}_2^{(0)} = U_2 \mathbf{e}_x$, respectively, in the X direction. $\mathbf{e}_x$ denotes the unit vector in the X direction. Let the interface between the fluids be at $z = 0$.

The mechanical energy of perfect fluids with irrotational flow, in presence of external conservative forces, can be described by the Bernoulli equation,

$$-\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \frac{p}{\rho} + \Phi = \mathcal{F}(t), \quad (\text{B1})$$

where $\mathcal{F}(t)$ is an arbitrary function of time, Φ is potential due to external fields, $\mathbf{u} = -\nabla \phi$, and ϕ is the velocity potential. The unperturbed velocity potential for lower fluid is $-U_1 x$. Since the fluids are perfect and incompressible, Kelvin's vorticity theorem assures that if the flow was irrotational to begin with, the velocities induced because of perturbations will also be irrotational.²⁸ Let the perturbed potential be

$$\phi_1 = \phi_1^{(0)} + \phi_1^{(1)} = -U_1 x + \phi_1^{(1)}. \quad (\text{B2})$$

Since the fluids are incompressible, $\Delta\phi_1 = 0$. Therefore, Eq. (B2) implies that $\Delta\phi_1^{(1)} = 0$. Likewise, if $\phi_2^{(1)}$ is the perturbed potential of the upper fluid, then $\Delta\phi_2^{(1)} = 0$. In terms of velocity potential of the lower fluid, the velocity of perturbed surface is

$$\dot{\xi}^{(1)} = -\frac{\partial\phi_1^{(1)}}{\partial z}, \quad (\text{B3})$$

where $\xi^{(1)}$ is the z coordinate of the perturbed interface, the overhead dot denotes the time derivative. Similar equation holds good for the upper fluid. An arbitrary perturbation of the surface and the corresponding velocity potentials can be expanded in terms of normal modes as

$$\xi^{(1)} = A \exp(nt + ikx), \quad (\text{B4})$$

$$\phi_1^{(1)} = C_1 \exp(nt + ikx + kz), \quad (\text{B5})$$

$$\phi_2^{(1)} = C_2 \exp(nt + ikx - kz). \quad (\text{B6})$$

The perturbations are written in a manner so that they vanish away from the interface. Substituting Eqs. (B4) to (B6) in (B3) and its analog for upper fluid, at $z = 0$:

$$-kC_1 = nA + ikU_1A, \quad (\text{B7})$$

$$kC_2 = nA + ikU_2A. \quad (\text{B8})$$

The third equation for finding the unknowns in Eqs. (B4) to (B6) follows from the continuity of pressure across the interface. Assuming a surface tension γ at the interface, we have at $z = 0$,

$$p_1 - \frac{E_1^2}{2}\rho_1 \frac{\partial\epsilon_1}{\partial\rho_1} + \frac{\epsilon_1}{2}E_1^2 = p_2 - \frac{E_2^2}{2}\rho_2 \frac{\partial\epsilon_2}{\partial\rho_2} + \frac{\epsilon_2}{2}E_2^2 - \gamma \frac{\partial^2\xi^{(1)}}{\partial x^2}, \quad (\text{B9})$$

where p_1 and p_2 are total pressure in lower and upper fluids, respectively. We can write it in simple form as

$$p_1 - K_{1e}^2 E_1^2 = p_2 - K_{2e}^2 E_2^2 - \gamma \frac{\partial^2\xi^{(1)}}{\partial x^2}, \quad (\text{B10})$$

where

$$K_{1e}^2 = \rho_1 \frac{\partial\epsilon_1}{\partial\rho_1} - \frac{\epsilon_1}{2}, \quad (\text{B11})$$

$$K_{2e}^2 = \rho_2 \frac{\partial\epsilon_2}{\partial\rho_2} - \frac{\epsilon_2}{2} \quad (\text{B12})$$

are constants depending only on the molecular and bulk properties of the fluids. For an incompressible fluid in a gravitational field,

$$\Phi_1 = g\xi^{(1)}. \quad (\text{B13})$$

Using Eqs. (B1) and (B13) in Eq. (B10),

$$\begin{aligned} \rho_1 \left(\frac{\partial\phi_1}{\partial t} - \frac{u_1^2}{2} - g\xi^{(1)} \right) - K_{1e}^2 E_1^2 &= \rho_2 \left(\frac{\partial\phi_2}{\partial t} - \frac{u_2^2}{2} - g\xi^{(1)} \right) - K_{2e}^2 E_2^2 + \\ &\quad \mathcal{F}(t) - \gamma \frac{\partial^2\xi^{(1)}}{\partial x^2}, \end{aligned} \quad (\text{B14})$$

where $\mathcal{F}(t) = \rho_2\mathcal{F}_2(t) - \rho_1\mathcal{F}_1(t)$. Under unperturbed conditions, with $u_1 = U_1$, $u_2 = U_2$, $\phi_1^{(1)} = 0$, $\phi_2^{(1)} = 0$, and $\xi^{(1)} = 0$, Eq. (B14) becomes

$$-\rho_1 \frac{U_1^2}{2} - K_{1e}^2 E^2(0^-) = -\rho_2 \frac{U_2^2}{2} - K_{2e}^2 E^2(0^+) + \mathcal{F}(t), \quad (\text{B15})$$

where $E(0^-)$ and $E(0^+)$ are the electric fields just below and above the interface. Substituting $\mathcal{F}(t)$ from the above equation in (B14), noting that $\mathbf{u}_1 = \mathbf{U}_1 + \mathbf{u}_1^{(1)}$, $\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{u}_2^{(1)}$, and using the linear (small amplitude) approximation,

$$\rho_1 \left[\frac{\partial \phi_1^{(1)}}{\partial t} - U_1 \frac{\partial \phi_1^{(1)}}{\partial x} - \check{g}_{1e} \xi^{(1)} \right] = \rho_2 \left[\frac{\partial \phi_2^{(1)}}{\partial t} - U_2 \frac{\partial \phi_1^{(1)}}{\partial x} - \check{g}_{2e} \xi^{(1)} \right] - \gamma \frac{\partial^2 \xi^{(1)}}{\partial x^2}, \quad (\text{B16})$$

where the “effective acceleration due to gravity” in each fluid is given by

$$\check{g}_{1e} = g + 2K_{1e}^2 \frac{E(0^-)}{\rho_1} \frac{\partial E_1}{\partial z} \quad (\text{B17})$$

and

$$\check{g}_{2e} = g + 2K_{2e}^2 \frac{E(0^+)}{\rho_2} \frac{\partial E_2}{\partial z}. \quad (\text{B18})$$

Using Eqs. (B4) to (B6) in (B16) we get the third equation in the unknowns A , C , and C' from which we get the dispersion relation

$$\frac{in}{k} = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{\rho_1 \check{g}_{1e} - \rho_2 \check{g}_{2e}}{k(\rho_1 + \rho_2)} + \frac{k\gamma}{\rho_1 + \rho_2} - \frac{\rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} \right]^{1/2}. \quad (\text{B19})$$

In absence of electric field $\check{g}_{1e} = \check{g}_{2e} = g$ and the dispersion relation reduces to the one in Lamb's treatise.²⁹ Thus the effect of an applied electric field gradient is to alter the specific weight $\rho \check{g}$. By choosing the value of the gradient $\partial E / \partial z$, we can effectively increase or decrease specific weight to our advantage.

We get the relations for magnetizing field gradients from Eqs. (B17) and (B18) by replacing ϵ_0 by μ_0 , κ_e by κ_m , K_e by K_m , and E by H . They are

$$\check{g}_{1m} = g + 2K_{1m}^2 \frac{H(0^-)}{\rho_1} \frac{\partial H_1}{\partial z} \quad (\text{B20})$$

and

$$\check{g}_{2m} = g + 2K_{2m}^2 \frac{H(0^+)}{\rho_2} \frac{\partial H_2}{\partial z}. \quad (\text{B21})$$

The constants K_{1m} and K_{2m} , given by Eqs. (B22) and (B23), depend only on the molecular and bulk properties of the fluids:

$$K_{1m}^2 = \rho_1 \frac{\partial \mu_1}{\partial \rho_1} - \frac{\mu_1}{2}, \quad (\text{B22})$$

$$K_{2m}^2 = \rho_1 \frac{\partial \mu_2}{\partial \rho_2} - \frac{\mu_2}{2}. \quad (\text{B23})$$

The dispersion relation, likewise, will be

$$\frac{in}{k} = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{\rho_1 \check{g}_{1m} - \rho_2 \check{g}_{2m}}{k(\rho_1 + \rho_2)} + \frac{k\gamma}{\rho_1 + \rho_2} - \frac{\rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} \right]^{1/2}. \quad (\text{B24})$$

The form of Eqs. (B17) and (B18) ((B20) and (B21)) tells that it is possible to choose magnetic (electric) fields gradients such that $\rho_1 \check{g}_{1m} = \rho_2 \check{g}_{2m}$ ($\rho_1 \check{g}_{1e} = \rho_2 \check{g}_{2e}$).

TABLE I. Physical properties of dielectric liquids.

Parameter	Water	Isobutylbenzene
Density (kg m^{-3})	1000	853
Dielectric constant	80	2.32
Breakdown voltage (kV/mm)	65	222

1. Electric and magnetic boundary conditions

I have not used the boundary conditions on the electric or magnetic fields in the analysis so far. They arise when we try to choose a field gradient such that the specific weights of the fluids match.

- Electric boundary condition: The relation $\rho_1 \check{g}_{1e} = \rho_2 \check{g}_{2e}$ gives

$$\rho_1 g + 2K_{1e}^2 E(0^-) E_{1,z} = \rho_2 g + 2K_{2e}^2 E(0^+) E_{2,z}.$$

If we assume that the electric field gradient is a constant G_e , we have

$$\rho_1 g + 2K_{1e}^2 E(0^-) G_e = \rho_2 g + 2K_{2e}^2 E(0^+) G_e.$$

The boundary condition on the normal component of the electric field is given by $(\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{n} = \sigma$,³⁰ where σ is the density of surface charges. If we assume the dielectrophoretic limit,³ that is there are no free surface charges, then we have $\kappa_{1e} E(0^-) = \kappa_{2e} E(0^+)$. Therefore, the field gradient can be chosen to be

$$G_e = \frac{(\rho_1 - \rho_2) \kappa_{2e} g}{2(\kappa_{1e} K_{2e}^2 - \kappa_{2e} K_{1e}^2) E(0^-)}.$$

- Magnetic boundary condition: The field gradient for magnetization field is chosen similarly except that the boundary condition $\kappa_{m1} H(0^-) = \kappa_{m2} H(0^+)$ is always true. It follows from the boundary condition $(\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} = 0$ or $(\mu_1 \mathbf{H}_1 - \mu_2 \mathbf{H}_2) \cdot \mathbf{n} = 0$.³⁰

2. Typical field strengths

I will estimate the electric field needed to achieve equal specific weight in isobutylbenzene and distilled water. Their physical parameters are listed in Table I.³¹

Isobutylbenzene being the lighter of the two is at the bottom and its parameters have a subscript “1.” Physical parameters of water have a subscript “2.” Using the Clausius-Mossotti relation of (A9) in Eqs. (B11) and (B12), we get $K_{1e}^2 = 4.431 \times 10^{-12}$ F/m and $K_{2e}^2 = 1.876 \times 10^{-8}$ F/m. If the electric field at the interface is set to 10^6 V/m, we get a value for gradient G_e to be -1.45×10^6 V/m². If the fluids are contained in a cell of half width $L = 1$ cm, and the voltage at the bottom boundary is set to zero, then that at the top boundary will be 1.97×10^4 V. (Electric field $E(z)$ is $E(0) + G_e z$, therefore, voltage is $V(z) = E_0 z + (1/2) G_e z^2$.) Voltage at the interface will be 9.93×10^3 V. Voltage difference between that at interface (top) and bottom (interface) is less than breakdown voltage of isobutylbenzene (water).

APPENDIX C: PROOFS OF THEOREMS

Theorem 1.

$$\int_{-\infty}^t G(t-t')(D^2 - k^2) w^{(1)} dt' = \eta^*(D^2 - k^2) w^{(1)}.$$

Proof. The normal mode expansion of $w^{(1)}$ is of the form,

$$w^{(1)} = \check{w}^{(1)}(z) \exp(nt + i\mathbf{k} \cdot \mathbf{x}).$$

Therefore, the left hand side (LHS) becomes

$$\begin{aligned} LHS &= \int_{-\infty}^t G(t-t')(D^2 - k^2)\check{w}^{(1)}(z) \exp(nt' + i\mathbf{k} \cdot \mathbf{x}) dt' \\ &= (D^2 - k^2)\check{w}^{(1)}(z) \exp(i\mathbf{k} \cdot \mathbf{x}) \int_{-\infty}^t G(t-t') \exp(nt') dt'. \end{aligned}$$

A transformation $s = t - t'$ simplifies the integral to

$$\int_{-\infty}^t G(t-t') \exp(nt') dt' = \exp(nt) \int_0^\infty G(s) \exp(-ns) ds = \exp(nt) \eta^*.$$

Therefore,

$$LHS = (D^2 - k^2)\check{w}^{(1)}(z) \exp(i\mathbf{k} \cdot \mathbf{x}) \exp(nt) \eta^* = \eta^* (D^2 - k^2) w^{(1)}. \quad (C1)$$

□

Theorem 2. For an eigenvalue n_i of equation (36) we have

$$(n_i^2 - (n_i^*)^2)I_3 = (n_i^* \eta - n_i \eta^*)I_4, \quad (C2)$$

where $I_3 > 0$, $I_4 > 0$ and

$$I_3 = \int_0^d \left\{ \rho^{(0)} |w_i^{(1)}|^2 + \frac{\rho^{(0)}}{k^2} |Dw_i^{(1)}|^2 \right\} dz, \quad (C3)$$

$$I_4 = \int_0^d \left\{ k^2 |w_i^{(1)}|^2 + 2 |Dw_i^{(1)}|^2 + \frac{|D^2 w_i^{(1)}|^2}{k^2} \right\} dz. \quad (C4)$$

Proof. One form of Eq. (36) is

$$\begin{aligned} &\check{g} \int_0^d w_j^{(1)} w_i^{(1)} D\rho^{(0)} dz - k^2 \gamma \int_0^d w_j^{(1)} w_i^{(1)} \delta(z - z^{(0)}) dz = \\ &\quad n_i^2 \int_0^d \left\{ \rho^{(0)} w_i^{(1)} w_j^{(1)} + \frac{\rho^{(0)}}{k^2} Dw_i^{(1)} Dw_j^{(1)} \right\} dz + \\ &\quad n_i \eta^* \int_0^d \left\{ k^2 w_i^{(1)} w_j^{(1)} + 2 Dw_i^{(1)} Dw_j^{(1)} + \frac{(D^2 w_i^{(1)})(D^2 w_j^{(1)})}{k^2} \right\} dz. \end{aligned} \quad (C5)$$

Interchanging i and j ,

$$\begin{aligned} &\check{g} \int_0^d w_j^{(1)} w_i^{(1)} D\rho^{(0)} dz - k^2 \gamma \int_0^d w_j^{(1)} w_i^{(1)} \delta(z - z^{(0)}) dz = \\ &\quad n_j^2 \int_0^d \left\{ \rho^{(0)} w_i^{(1)} w_j^{(1)} + \frac{\rho^{(0)}}{k^2} Dw_i^{(1)} Dw_j^{(1)} \right\} dz + \\ &\quad n_j \eta^* \int_0^d \left\{ k^2 w_i^{(1)} w_j^{(1)} + 2 Dw_i^{(1)} Dw_j^{(1)} + \frac{(D^2 w_i^{(1)})(D^2 w_j^{(1)})}{k^2} \right\} dz. \end{aligned} \quad (C6)$$

If $n_j = n_i^*$, then

$$\begin{aligned} & \check{g} \int_0^d |w_i^{(1)}|^2 D\rho^{(0)} dz - k^2 \gamma \int_0^d |w_i^{(1)}|^2 \delta(z - z^{(0)}) dz = \\ & (n_i^*)^2 \int_0^d \left\{ \rho^{(0)} |w_i^{(1)}|^2 + \frac{\rho^{(0)}}{k^2} |Dw_i^{(1)}|^2 \right\} dz + \\ & n_i^* \eta \int_0^d \left\{ k^2 |w_i^{(1)}|^2 + 2 |Dw_i^{(1)}|^2 + \frac{|D^2 w_i^{(1)}|^2}{k^2} \right\} dz. \end{aligned} \quad (C7)$$

Under the same assumptions, Eq. (C5) becomes

$$\begin{aligned} & \check{g} \int_0^d |w_i^{(1)}|^2 D\rho^{(0)} dz - k^2 \gamma \int_0^d |w_i^{(1)}|^2 \delta(z - z^{(0)}) dz = \\ & (n_i)^2 \int_0^d \left\{ \rho^{(0)} |w_i^{(1)}|^2 + \frac{\rho^{(0)}}{k^2} |Dw_i^{(1)}|^2 \right\} dz + \\ & n_i \eta^* \int_0^d \left\{ k^2 |w_i^{(1)}|^2 + 2 |Dw_i^{(1)}|^2 + \frac{|D^2 w_i^{(1)}|^2}{k^2} \right\} dz. \end{aligned} \quad (C8)$$

Subtracting Eq. (C7) from Eq. (C8),

$$\begin{aligned} & (n_i^2 - (n_i^*)^2) \int_0^d \left\{ \rho^{(0)} |w_i^{(1)}|^2 + \frac{\rho^{(0)}}{k^2} |Dw_i^{(1)}|^2 \right\} dz = \\ & (n_i^* \eta - n_i \eta^*) \int_0^d \left\{ k^2 |w_i^{(1)}|^2 + 2 |Dw_i^{(1)}|^2 + \frac{|D^2 w_i^{(1)}|^2}{k^2} \right\} dz. \end{aligned} \quad (C9)$$

The integrands in the above equation are positive definite and so are the limits, therefore the integrals themselves are positive. We can write Eq. (C9) in a simpler form as

$$(n_i^2 - (n_i^*)^2) I_3 = (n_i^* \eta - n_i \eta^*) I_4, \quad (C10)$$

where $I_3 > 0$, $I_4 > 0$ and

$$I_3 = \int_0^d \left\{ \rho^{(0)} |w_i^{(1)}|^2 + \frac{\rho^{(0)}}{k^2} |Dw_i^{(1)}|^2 \right\} dz, \quad (C11)$$

$$I_4 = \int_0^d \left\{ k^2 |w_i^{(1)}|^2 + 2 |Dw_i^{(1)}|^2 + \frac{|D^2 w_i^{(1)}|^2}{k^2} \right\} dz. \quad (C12)$$

□

Theorem 3. For an eigenvalue n_i of Eq. (36) we have

$$(n_i^* - n_i) I_5 + \left(\frac{\check{g}}{n_i} - \frac{\check{g}}{n_i^*} \right) I_6 - \left(\frac{k^2 \gamma}{n_i} - \frac{k^2 \gamma}{n_i^*} \right) I_7 = 0, \quad (C13)$$

where

$$I_5 = \int_0^d \left\{ \rho^{(0)} |w_i^{(1)}|^2 + \frac{\rho^{(0)}}{k^2} |Dw_i^{(1)}|^2 \right\} dz > 0, \quad (\text{C14})$$

$$I_6 = \int_0^d |w_i^{(1)}|^2 D\rho^{(0)} dz, \quad (\text{C15})$$

$$I_7 = \int_0^d |w_i^{(1)}|^2 \delta(z - z^{(0)}) dz > 0. \quad (\text{C16})$$

Proof. Proof of Theorem 2 involved manipulating a form of Eq. (36). If we continue with that equation in its original form and interchange i and j , we get

$$\begin{aligned} -n_j \int_0^d \left\{ \rho^{(0)} w_j^{(1)} w_i^{(1)} + \frac{\rho^{(0)}}{k^2} Dw_j^{(1)} Dw_i^{(1)} \right\} dz + \frac{\check{\gamma}}{n_j} \int_0^d w_i^{(1)} w_j^{(1)} D\rho^{(0)} dz \\ - \frac{k^2 \gamma}{n_j} \int_0^d w_i^{(1)} w_j^{(1)} \delta(z - z^{(0)}) dz = \\ \eta^* \int_0^d \left\{ k^2 w_j^{(1)} w_i^{(1)} + 2Dw_j^{(1)} Dw_i^{(1)} + \frac{x}{k^2} (D^2 w_j^{(1)})(D^2 w_i^{(1)}) \right\} dz. \end{aligned} \quad (\text{C17})$$

Subtracting it from Eq. (36), we get

$$\begin{aligned} (n_j - n_i) \int_0^d \left\{ \rho^{(0)} w_j^{(1)} w_i^{(1)} + \frac{\rho^{(0)}}{k^2} Dw_j^{(1)} Dw_i^{(1)} \right\} dz + \\ \left(\frac{\check{\gamma}}{n_i} - \frac{\check{\gamma}}{n_j} \right) \int_0^d w_i^{(1)} w_j^{(1)} D\rho^{(0)} dz - \\ \left(\frac{k^2 \gamma}{n_i} - \frac{k^2 \gamma}{n_j} \right) \int_0^d w_i^{(1)} w_j^{(1)} \delta(z - z^{(0)}) dz = 0. \end{aligned} \quad (\text{C18})$$

If we now choose $n_j = n_i^*$, we can write Eq. (C18) as

$$(n_i^* - n_i) I_5 + \left(\frac{\check{\gamma}}{n_i} - \frac{\check{\gamma}}{n_i^*} \right) I_6 - \left(\frac{k^2 \gamma}{n_i} - \frac{k^2 \gamma}{n_i^*} \right) I_7 = 0, \quad (\text{C19})$$

where

$$I_5 = \int_0^d \left\{ \rho^{(0)} |w_i^{(1)}|^2 + \frac{\rho^{(0)}}{k^2} |Dw_i^{(1)}|^2 \right\} dz > 0, \quad (\text{C20})$$

$$I_6 = \int_0^d |w_i^{(1)}|^2 D\rho^{(0)} dz, \quad (\text{C21})$$

$$I_7 = \int_0^d |w_i^{(1)}|^2 \delta(z - z^{(0)}) dz > 0. \quad (\text{C22})$$

The sign of I_6 depends on the density gradient. It is positive if density increases with increasing z and is negative if density decreases with increasing z . We can simplify (C19) to get

$$(n_i^* - n_i) \left(I_5 + \frac{\check{g} I_6}{|n_i|^2} - \frac{k^2 \gamma I_7}{|n_i|^2} \right) = 0. \quad (\text{C23})$$

□

1. Descartes rule of signs

Descartes rule of signs³² is

1. If the terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Multiple roots of the same value are counted separately.
2. As a corollary of the point 1, the number of negative roots is the number of sign changes after multiplying the coefficients of odd-power terms by 1, or fewer than it by a multiple of 2.

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