GENERALIZED RECURRENT AND CONCIRCULAR RECURRENT MANIFOLDS

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The concepts of generalized recurrent manifold $GK_n$ and generalized concircular recurrent manifolds $G(ZK_n)$ are considered to prove that a manifold $(M_n, g)$ which is either $GK_n$ or $G(ZK_n)$ is concircular recurrent manifold $ZK_n$. By using result due to Pushpa Desai and Krishna Amr 

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1. PRELIMINARIES

Let $(M_n, g)$ be a Riemannian manifold of dimension $n$ with metric tensor $g_{ij}$. We denote $\nabla_i R^k_{jl}$, $R_{ij}$, and $R$ the operator of covariant differentiation with respect to $g_{ij}$, the curvature tensor, the Ricci tensor and the scalar curvature of $M$ respectively.

For a manifold $(M_n, g)$ of constant curvature the curvature tensor is given by Yano's

$$R^k_{jl} = \frac{R}{n(n-1)} \left( \delta^k_j g_{li} - \delta^k_l g_{ji} \right). \quad \ldots \quad (1.1)$$

The tensor $G_{ji}$ of type $(0, 2)$, called Einstein tensor, is given by

$$G_{ji} = R_{ji} - \frac{R}{n} g_{ji}. \quad \ldots \quad (1.2)$$

If $G_{ji} = 0$ or $R_{ji} = \frac{R}{n} g_{ji}$, then $(M_n, g)$ is called Einstein space.

If $(M_n, g)$ admits a tensor $T$ of type $(p, q)$ with components $T^i_{\mu_1 \ldots \mu_p}$ satisfying
\[ \nabla_1 T_{j_1 \ldots j_q}^i = K_{j_1 \ldots j_q}^i \]  
\[ \ldots (1.3) \]

for some non-zero vector field \( K \) with components \( K_i \), then \( M_n \) is called \( T \)-recurrent and is denoted by \( T_n \). If \( \nabla_1 T_{j_1 \ldots j_q}^i = 0 \), then \( M_n \) is called \( T \)-symmetric. If \((M_n, g)\) is recurrent with respect to curvature tensor \( R_{kli}^b \), then it is called recurrent space and denoted by \( K_n \). If it is symmetric with respect to curvature tensor, then it is called symmetric space.

In a recent paper, De and Guha\(^1\) have introduced a non-flat \((M_n, g)\), whose curvature tensor \( R(X, Y, Z) \) satisfies the condition

\[ (\nabla g) R(X, Y, Z) = A(U) R(X, Y, Z) + B(U) [g(Y, Z)X - g(X, Z)Y] \]  
\[ \ldots (1.4) \]

where \( A \) and \( B \) are 1-forms and \( P, Q \) are two vector fields satisfying,

\[ g(X, P) = A(X) \]

and

\[ g(X, Q) = B(X). \]  
\[ \ldots (1.5) \]

Such a manifold is called a generalized recurrent manifold and is denoted by \( GK_n \) and the 1-form \( B \) is called its associated form. Note that when \( B = 0 \), \((M_n, g)\) is recurrent space \( K_n \).

Further a non-flat Riemannian manifold \((M_n, g)\) is called generalized concircular recurrent manifold \( G(ZK_n) \), if its concircular curvature tensor

\[ Z(X, Y, Z) = R(X, Y, Z) - \frac{R}{n(n-1)} [g(Y, Z)X - g(X, Z)Y] \]  
\[ \ldots (1.6) \]

satisfies the condition

\[ (\nabla g) Z(X, Y, Z) = A(U) Z(X, Y, Z) + B(U) (g(Y, Z)X - g(X, Z)Y) \]  
\[ \ldots (1.7) \]

where \( A, B \) and \( V \) are as mentioned above.

U. C. De and D. Kamily\(^2\) have proved the following results:

**Theorem A** --- A necessary and sufficient condition for a \( G(ZK_n) \) to be \( GK_n \) is that

\[ dr(X) = r A(X) = 0. \]  
\[ \ldots (1.8) \]

**Theorem B** --- In a \( G(ZK_n) \) the scalar curvature is constant if and only if

\[ A(X) + (n-1)B(X) = 0. \]  
\[ \ldots (1.9) \]

Pushpa Desai and Krishna Amrut\(^3\) have proved

**Theorem C** --- A \( ZK_n \) is either \( K_n \) or Einstein
2. THEOREMS

**Theorem 2.1** — A Riemannian manifold \((M_n, g)\) which is either \(G(ZK_n)\) or \(GK_n\) is concircular recurrent manifold \(ZK_n\).

**Proof:** Suppose \((M_n, g)\) is \(G(ZK_n)\). Then by expressing (1.7) in components, we have

\[
\nabla_l Z^h_{kji} = A_l Z^h_{kji} + B_l \left( \delta^h_k g_{ji} - \delta^h_j g_{ki} \right). \tag{2.1}
\]

By contracting (2.1) with respect to \(h\) and \(k\) and using (1.2) we get

\[
\nabla_l G_{ji} = A_l G_{ji} + (n - 1)B_l g_{ji}. \tag{2.2}
\]

Transvecting (2.2) with \(g^{ji}\) and noting \(g^{ji} G_{ji} = 0\), we get \(B_1 = 0\). So from (2.1) we get

\[
\nabla_l Z^h_{kji} = A_l Z^h_{kji}. \tag{2.3}
\]

Now suppose \((M_n, g)\) is \(GK_n\) so that (1.4) holds. The contraction of (1.4) written in components form with respect to \(h\) and \(k\) gives

\[
\nabla_l R_{ji} = A_l R_{ji} + (n - 1)B_l g_{ji}. \tag{2.4}
\]

and hence

\[
\nabla_l R = A_l R + n(n - 1)B_l. \tag{2.5}
\]

On substituting (1.6), written in components form, in RHS of (2.1) we get

\[
\nabla_l Z^h_{kji} = \nabla_l R^h_{kji} - \frac{\nabla_l R}{n(n-1)} \left[ \delta^h_k g_{ji} - \delta^h_j g_{ki} \right]. \tag{2.6}
\]

which by using (1.4) written in components form gives

\[
\nabla_l Z^h_{kji} = A_l Z^h_{kji} + B_l \left[ \delta^h_k g_{ji} - \delta^h_j g_{ki} \right] - \frac{\nabla_l R}{n(n-1)} \left[ \delta^h_k g_{ji} - \delta^h_j g_{ki} \right]. \tag{2.7}
\]

Expressing \(R^h_{kji}\) in terms of \(Z^h_{kji}\) by using (1.6) written in components form and substituting the result in (2.7), we get

\[
\nabla_l Z^h_{kji} = A_l Z^h_{kji} + \left[ \frac{A_l R - \nabla_l R + n(n-1)B_l}{n(n-1)} \right] \left[ \delta^h_k g_{ji} - \delta^h_j g_{ki} \right]. \tag{2.8}
\]

Now by virtue of (2.5) it shows that \(\nabla_l Z^h_{kji} = A_l Z^h_{kji}\).
In view of theorem C, the following theorem can be stated.

**Theorem 2.2** — A Riemannian manifold \((M_n, g)\) which is either \(G(ZK_n)\) or \(GK_n\) is \(K_n\) or Einstein space.

**Proof** : By Theorem 2.1 \((M_n, g)\) is \(ZK_n\). Now by applying theorem C, the result is obtained.

**Theorem 2.3** — A \(G(ZK_n)\) is recurrent manifold \(K_n\) if and only if the scalar curvature \(R = 0\).

**Proof** : From theorem 2.1, a \(G(ZK_n)\) is \(ZK_n\) and hence (1.4) holds, which in view of (1.6) written in components form gives

\[
\left( \nabla_i R_{kji} - A_f R^f_{kji} \right) = \frac{\nabla R_i - A R_i}{n(n-1)} \left[ \delta^h_{k} g_{ji} - \delta^h_{j} g_{ki} \right]. \tag{2.9}
\]

The contraction of (1.4), written in components form, with respect to \(h\) and \(k\) gives

\[
\nabla_i G_{ji} = A_f G_{ji} \tag{2.10}
\]

Now by using the identity

\[
\nabla_i Z^h_{kji} + \nabla_k Z^h_{jli} + \nabla_j Z^h_{lki} = 0
\]

we get

\[
A_i Z^h_{kji} + A_k Z^h_{jli} + A_j Z^h_{lki} = 0. \tag{2.11}
\]

Contracting (2.11) with respect to \(h\) and \(k\) and using the property

\[
Z^h_{kji} = -Z^h_{jki}
\]

we get

\[
(A_f G_{ji} - A_j G_{fi}) + A_i \nabla Z^h_{jki} = 0,
\]

which by transvecting with \(g^{li}\), gives

\[
A_i G'_{ji} = 0. \tag{2.12}
\]

By virtue of (2.10) the above result (2.12) gives \(\nabla_i G'_{ji} = 0\) and hence

\[
\nabla_i R'_{ji} - \frac{\nabla R}{n} = 0. \tag{2.13}
\]

By using the identity \(\nabla_i R'_{ji} = \frac{1}{2} \frac{\nabla R}{n}\) in (2.13) we get,
\[ \nabla_i R = 0 \]  \hspace{1cm} \text{(2.14)}

Now from (2.9) we get,

\[ \left( \nabla_i R^h_{kji} - A^h_{kji} \right) = -\frac{A^h_{kji}}{n(n-1)} \left[ \delta^h_{kji} - \delta^h_{jki} \right] \]  \hspace{1cm} \text{(2.15)}

which proves the theorem.

**Theorem 2.4** — In a $G(ZK_n)$ the scalar curvature is constant.

**Proof**: If $(M_n, g)$ is $G(ZK_n)$, then by theorem 2.1 it is $ZK_n$ also, i.e. (1.4) holds. As in the proof of the theorem 2.3, we get, $\nabla_i R = 0$.

**Remark 2.1** — In theorem 2, the authors have proved that $G(ZK_n)$ is of constant scalar curvature under the condition (1.9). In the proof of the above theorem we see that the condition (1.9) is not required.

**Theorem 2.5** — $G(ZK_n)$ with non zero scalar curvature is Einstein space.

**Proof**: From theorem 2.1, a $G(ZK_n)$ is $ZK_n$. But by theorem 2, $ZK_n$ is either $K_n$ or Einstein. If $R \neq 0$, then (2.15) leads to the fact that the space is not recurrent. Hence, it must be Einstein.

**References**