A special form of Rund’s h-curvature tensor using $R3$-like Finsler space

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Abstract
The purpose of the present paper is to consider and study a special form of Rund’s h-curvature tensor $K^i_ljk$ and Berwald’s curvature tensor $H^i_ljk$ in an $R3$-like C-reducible Finsler space. In this paper, we modify the Rund’s h-curvature tensor $K^i_ljk$ to special form by using some special Finsler spaces like C-reducible, $R3$-like Finsler spaces.

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1 Introduction
Let $F^n = (M^n, F)$ be an $n$-dimensional Finsler space with the fundamental function $F(x, y)$. The Finsler $\Gamma_v$-connection $R^\Gamma$, constructed from the Cartan connection $C^\Gamma$, is called the Rund connection. Matsumoto defined the curvature tensor $R^\Gamma$ and the concept of special form of Rund’s curvature tensor $K^i_ljk$ [5]. The author [9] have studied Finsler space with Rund’s h-curvature tensor $K^i_ljk$ of a special form. Here we extend the study of a special form of Rund’s h-curvature tensor using $R3$-like Finsler space and obtain some results. In this paper, the range of indexes varies from 1 to $n$ and the $\nu$-covariant and $\sigma$-covariant derivatives are denoted by $|j$ and $\parallel j$, respectively.

We use the following notations from [5, 8]:

(a) $g_{ij} = \frac{1}{2} \partial^i \partial^j L^2$, $g^{ij} = (g_{ij})^{-1}$, $\partial^i = \frac{\partial}{\partial y^i}$
(b) $C_{ijk} = \frac{1}{2} \partial^k g_{ij}$, $C^i_{ij} = \frac{1}{2} g^{km} (\partial^i g_{mj})$
(c) $h_{ij} = g_{ij} - l_i l_j$, $h^i_j = \delta^i_j - l^i l_j$
(d) $C^i_{tr} y^r = 0$, $h^i_j y^j = 0$
(e) $y^i L_k = y^i_k L$, $g_{ij} y^i y^j = F^2$, $l_i y^i = F$
(f) $L_k = y_k L$

Definition 1.1 (see [1, 2]). A Finsler space $F^n (n > 3)$ is called $R3$-like, if the curvature tensor $R_{hijk}$ is written in the form

$$R_{hijk} = g_{hj} L_{ik} + g_{ik} L_{hj} - g_{hk} L_{ij} - g_{ij} L_{hk}$$

(1.2)

where $L = L_{ij} y^j$ is the tensor.
Definition 1.2 (see [2, 3]). A Finsler space $F^n$ is called C-reducible if it satisfies the equation

$$C_{ijk} = \left( C_i h_{jk} + C_j h_{ki} + C_k h_{ij} \right) / (n + 1)$$  \hspace{1cm} (1.3)

where $C_i = C_{ijk} g^{jk}$.

Definition 1.3 (see [6]). A Finsler space $F^n$ is called P-reducible if the torsion tensor $P_{ijk}$ is written as

$$P_{ijk} = G_{i}^{h} h_{jk} + G_{j}^{h} h_{ki} + G_{k}^{h} h_{ij}$$  \hspace{1cm} (1.4)

where $P_{ijk} = C_{ijk} |_{0}$ and $G_{i}^{h} = C_{i|0}^{h} / (n + 1)$.

The $u$-covariant derivative of P-reducible Finsler space is given by [9]

$$P_{ij|k} = G_{i}^{h} h_{jk} + G_{j}^{h} h_{ki} + G_{k}^{h} h_{ij}$$  \hspace{1cm} (1.5)

Definition 1.4 (see [7]). A Finsler space is called Landsberg Finsler space if $C_{ijk}|_{0} = 0$.

We use the following identities from [5, 7, 9]:

(a) $R_{ijk|l} = K_{ijk}^{l} + U_{(ijk)} \{ P_{jr}^{l} P_{kl}^{r} + P_{klj}^{l} \} = 0$

(b) $R_{ijk}^{l} y^{l} = R_{ijk}^{l} = R_{ijkl}^{l}$

(c) $K_{ijk}^{l} + K_{jkl}^{i} + K_{klj}^{i} = 0$

(d) $H_{ijk}^{l} = R_{ijk|l}$

(e) $H_{ijk}^{l} = H_{jkl}^{l} + H_{kjl}^{l} = 0$

(f) $K_{ijk}^{l} = - K_{ikj}^{l}$, $K_{ijk}^{l} = g_{r} K_{ijk}^{l}$

(g) $h_{ijk|k} = 2 C_{ijk} - L^{-2} \left( y_{i} h_{jk} + y_{j} h_{ik} \right)$

(h) $h_{ijk|k}^{l} = - L^{-2} \left( y_{j} h_{ik}^{l} + y_{i} h_{jk}^{l} \right)$

where $R_{ijk}^{l}$ is the $(v)$ h-torsion tensor and the suffix ‘0’ means contraction with $y^{l}$. The notation $u_{(ijk)}$ denotes the interchange on indices $j$ and $k$ and substraction.

We use the following lemma in the next section.

Lemma 1.5 (see [4]). If the equation $v_{k} h_{ij} + v_{i} h_{jk} + v_{j} h_{ik} = 0$ holds in $F^n$, then we have (1) $v_{ij} = 0$, $(n \geq 4)$ and (2) $v_{ij} = v(m_{i} n_{j} - m_{j} n_{i})$, with reference to the Moor frame $(l^{i}, m^{i}, n^{i})$, where $v$ is a scalar.

2 Special form of Rund’s h-curvature tensor $K_{ijk}^{l}$

Let $F^{n}$ be a Finsler space with Rund’s h-curvature tensor $K_{ijk}^{l}$ of the special form [9]

$$K_{ijk}^{l} = U_{(ijk)} \{ A_{jk} h_{i}^{l} + D_{ik} h_{j}^{l} + E_{j}^{l} h_{ki} \}$$  \hspace{1cm} (2.1)

where $A_{jk}$, $D_{ik}$, $E_{j}^{l}$, $F_{j}^{l}$, $G_{l}$ are Finsler tensor fields.

Consider h-curvature tensor of the form

$$K_{ijk}^{l} = R_{ijk}^{l} - C_{l}^{i} R_{ijk}^{l}$$  \hspace{1cm} (2.2)
Using equations (1.2) and (1.3) in (2.2), we get
\[
K^i_{jk} = \left\{ \delta^i_j L_{ik} + g_{ik} L^i_j - \delta^i_k L_{ij} - g_{ij} L^i_k \right\} \\
- \left( \delta^i_j L_{ik} + y_k L^i_j - \delta^i_k L_{ij} - y_j L^i_k \right) \left( C^i h_{tr} + C_i h^r + C_r h^i_k \right) / (n+1) \}
\]

By using some Finsler identities, the above equation can be written as
\[
K^i_{jk} = \left\{ (h^i_j + l^i_l j) L_{ik} + (h_{ik} + l^i_l k) L^i_j - (h^i_j + l^i_l j) L^i_k \right\} \\
- \left\{ C^i h_{tr} \delta^i_j L_{ik} + C_i h^r \delta^i_j L_{ik} + C_r h^i_k \delta^i_j L_{ik} + C^i h_{tr} y_k L^i_j C_i h^r y_k L^i_j + C_r h^i_k y_k L^i_j \right\} \\
- \left\{ C^i h_{tr} \delta^i_k L_{ij} - C_i h^r \delta^i_k L_{ij} - C_r h^i_k \delta^i_k L_{ij} - C^i h_{tr} y_j L^i_k - C_i h^r y_j L^i_k - C_r h^i_k y_j L^i_k \right\} / (n+1)
\]

After simplification and the rearrange the terms, we get
\[
K^i_{jk} = U_{(jk)} \left[ h^i_j (2C_k L_j)/(n+1) + h^i_j L_{ik} L_j - 2C_i L_j y_k / (n+1) \right] \\
+ h_{ik} \left( L^i_j + L^i_l j + 2C^i L_y j / (n+1) \right)
\]

In simple form, the above equation can be written as a special form of (2.1) as
\[
K^i_{jk} = U_{(jk)} \left( h^i_j A_{jk} + h^i_j D_{ik} + h_{ik} E^i_j \right)
\]

where
\[
A_{jk} = 2C_k L_j / (n+1) \\
D_{ik} = (L_{ik} + L^i_l k - 2C_i L y_k / (n+1)) \\
E^i_j = (L^i_j + L^i_l l + 2C^i L y_j / (n+1))
\]

Thus we state the following.

**Theorem 2.1.** In an R3-like, C-reducible Finsler space, the h-curvature tensor reduces to special form of Rund’s h-curvature tensor (2.3).

Now we compare the Rund’s curvature tensor and h-curvature tensor. Thus, from (2.1) and (2.2), we have
\[
R^i_{jk} - C^i_k R^r_{jk} = \left\{ A_{jk} h^i_j + D_{ik} h^i_j + E^i_j h_{kl} - A_{kj} h^i_l - D_{lj} h^i_k - E^i_j h_{jl} \right\}
\]

Contracting (2.5) with respect to \( y^l, y^k \) and using (1.1d), we get
\[
R^i_{j0} = D_{00} h^i_j
\]

Again contracting (2.6) with respect to \( i \) and \( j \), we get
\[
R = (n - 1) D_{00}
\]

Now, we will find \( D_{00} \). Consider \( D_{ij} \) from the special form (2.3), and contract this with respect to \( i \) and \( j \), and by using (1.1e), we have
\[
D_{00} = 2LF^2
\]

Substituting (2.8) in (2.7), we have
\[
R = 2(n-1)LF^2
\]

Thus we state the following.
**Theorem 2.2.** If the Rund’s h-curvature tensor has the special form (2.1), then the scalar curvature of the space is $2(n - 1)LF^2$.

Let us suppose that $F^n$ is $R3$-like C-reducible Finsler space. Then, by using (1.3) and (2.3), the h-curvature tensor (2.2) can written as

$$R_{ijk}^l = K_{ijk}^l + C_{ik}^r R_{jk}^r$$

$$R_{ijk}^l = U_{ijk} \{ A_{jk} h_i^l + D_{ik} h_j^l + E_{ij}^l h_{kl} \}$$

$$R_{ijk}^l = U_{ijk} \{ A_{jk} + (2C_j L_{yk})/(n + 1) \} h_i^l - \{ A_{kj} + (2C_k L_{ij})/(n + 1) \} h_i^j$$

$$R_{ijk}^l = U_{ijk} \{ A_{jk} + (2C_j L_{yk})/(n + 1) \} h_i^l - \{ A_{kj} + (2C_k L_{ij})/(n + 1) \} h_i^k$$

$$R_{ijk}^l = U_{ijk} \{ A_{jk} + (2C_j L_{yk})/(n + 1) \} h_i^l - \{ A_{kj} + (2C_k L_{ij})/(n + 1) \} h_i^k$$

(2.9)

$$R_{ijk}^l = U_{ijk} \{ Q_{jk} h_i^l + N_{lk} h_j^k + M_j^l h_{kl} \}$$

where

$$Q_{jk} = \{ A_{jk} + 2C_j L_{yk}/(n + 1) \}$$

$$N_{lk} = \{ D_{lk} + 2C_l L_{yk}/(n + 1) \}$$

$$M_j^l = \{ E_j^l - 2C^l L_{yj}/(n + 1) \}$$

Thus we have the following.

**Theorem 2.3.** In an $R3$-like C-reducible Finsler space, if the Rund’s h-curvature tensor has the special form (2.3), then the Cartan h-curvature tensor $R_{ijk}^l$ has the special form (2.9).

Using the special form of Rund’s h-curvature tensor $K_{ijk}^l$ in the Bianchi identity (1.6d), we get

$$\left( A_{jk} - A_{kj} + D_{kj} - D_{jk} \right) h_i^l + \left( A_{lk} - A_{lk} + D_{lk} - D_{kl} \right) h_j^l$$

$$+ \left( A_{ij} - A_{jl} + D_{jl} - D_{ij} \right) h_k^l = 0$$

(2.10)

Due to Lemma 1.1, equation (2.10) can be written as

$$A_{jk} - A_{kj} = D_{jk} - D_{kj}$$

Thus we have the following.

**Theorem 2.4.** If the Rund’s h-curvature tensor $K_{ijk}^l$ is of the special form (2.3), then both the tensor fields $A_{ij}$ and $D_{ij}$ are symmetric simultaneously.

It is also known that a Finsler space is Landsberg space with $P_{ijk} = C_{ijk}/0 = 0$. If $F^n$ is Landsberg, then from (1.6a) and (1.6e), we get

$$R_{ijk}^{l|l} - K_{ijk}^l = 0 \text{ or } H_{ijk}^l = K_{ijk}^l$$

Thus we can propose the following.
Corollary 2.5. If $F^n$ is a Landsberg space and the Rund’s h-curvature tensor $K^i_{ijk}$ is of the form (2.3), then Cartan curvature tensor coincides with the Berwald’s curvature tensor.

Now consider h-curvature tensor (1.6a) of the form

$$R^i_{jk|l} = K^i_{ijk} - U_{(ijk)} \{ P^i_{jr} P^r_{kl} + P^i_{klj} \}$$

From equation (1.6e), the above equation can be written as

$$H^i_{jk} = K^i_{ijk} - U_{(ijk)} \{ P^i_{jr} P^r_{kl} + P^i_{klj} \}$$

Suppose $F^n$ is a P-reducible Finsler space, then by using (1.4), (1.5), (2.3), and (2.11), we have

$$H^i_{jk} = \{ A_{jk} h^i_j + D_{lk} h^i_k + E^i_{jk} h_{kl} - A_{kj} h^i_l - D_{lj} h^i_k - E^i_{lk} h_{jl} \}$$

$$H^i_{jk} = \{ A_{jk} h^i_j + D_{lk} h^i_k + E^i_{jk} h_{kl} - A_{kj} h^i_l - D_{lj} h^i_k - E^i_{lk} h_{jl} \} - U_{(ijk)} \{ (G^2 G h_{jr} + G^2 G h_{jl}) (G^2 G h_{kl} + G^2 G h_{lj}) + (G G_{kl} h^i_j + G G_{lj} h^i_k + G G_{jk} h^i_l) \}$$

$$H^i_{jk} = \{ A_{jk} h^i_j + D_{lk} h^i_k + E^i_{jk} h_{kl} - A_{kj} h^i_l - D_{lj} h^i_k - E^i_{lk} h_{jl} \} - (G G_{kj} h^i_l + G G_{lj} h^i_k + G G_{jk} h^i_l) - (G^2 G h_{jr} + G^2 G h_{jl}) - (G^2 G h_{kl} + G^2 G h_{lj}) - (G^2 G h_{kl} + G^2 G h_{lj})$$

$$H^i_{jk} = \{ T^i_{jk} h^i_j + M^i_{lk} h^i_k + N^i_j h_{kl} \}$$

where

$$T^i_{jk} = A_{jk} - g_{kl}$$

$$M^i_{lk} = D_{lk} + G_{lj} G^i_k - G G^i_r h_{jl} - G G^i_k h_{lj} - G G^i_l h_{jk}$$

$$N^i_j = E^i_j - G^i_j G^r h_{jr} - G^i_j G^r h_{jr} + G^i_j G^r h_{jr}$$

Thus we have the following.

Theorem 2.6. In a P-reducible Finsler space, and the special form of Rund’s h-curvature tensor $K^i_{ijk}$ has the special form of Berwald’s curvature tensor, then $H^i_{jk}$ is of the form (2.12).

Consider the Bianchi identity

$$H^i_{jk} + H^i_{kj} + H^i_{lj} = 0$$

Substituting (2.12) in (2.13), we get

$$[(T_{jk} - T_{kj} + M_{kj} - M_{jk}) h^i_j + (T_{kl} - T_{lk} + M_{lk} - M_{kl}) h^i_k$$

$$+ (T_{lj} - T_{jl} + M_{lj} - M_{jl}) h^i_l] = 0$$

Due to Lemma 1.1, equation (2.14) can be written as

$$T_{jk} - T_{kj} = M_{jk} - M_{kj}$$

Thus we have the following.
Theorem 2.7. If in a Finsler space $F^n$ the h-curvature tensor $H_{ijkl}^i$ is of the form (2.12), then the tensor fields $T_{jk}$ and $M_{kj}$ both are simultaneously symmetric.

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References


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