On N(k)-Mixed Quasi Einstein Manifolds

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Abstract. In this paper N(k)-Mixed Quasi Einstein Manifolds \((N(k) - (MQE))\) are introduced and the existence of these manifolds is proved. We give hyper surfaces of Euclidean spaces as examples of \((N(k) - (MQE))\) manifolds and semi symmetric, ricci symmetric and ricci recurrent \((N(k) - (MQE))\) manifolds are studied.

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1. Introduction

M.C.Chaki and R.K.Maity [1] introduced the concept quasi Einstein manifolds. A non-flat Riemannian manifold \((M^n, g)(n > 2)\) is said to be a quasi Einstein manifold if its ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the condition

\[ S(X, Y) = a g(X, Y) + bA(X)A(Y), \]

where \(a\) and \(b\) are smooth functions of which \(b \neq 0\) and \(A\) is a non zero 1-form such that \(g(X, U) = A(X)\), for all vector fields \(X\) and \(U\) is a unit vector field. U.C.De and Gopal Chandra Ghosh [4, 5] generalized the quasi Einstein manifolds. A non-flat Riemannian manifold \((M^n, g)(n > 2)\) is said to be a generalized quasi Einstein manifold if its ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the condition

\[ S(X, Y) = a g(X, Y) + bA(X)A(Y) + cB(X)V(Y), \]

where \(a\), \(b\) and \(c\) are certain smooth functions, \(A\) and \(B\) are non zero 1-forms, and \(U\) and \(V\) are unit vector fields corresponding to 1-forms \(A\) and \(B\) respectively such that \(g(X, U) = A(X)\), \(g(X, V) = B(X)\) and \(g(U, V) = 0\). The vector fields \(U\) and \(V\) are called generators of...
the quasi Einstein manifold. The $k$-nullity distribution $N(k)$ [8] of a Riemannian manifold $M$ is defined by

$$N(k): p \rightarrow N_p(k) = \{Z \in T_p M | R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}$$

for all $X,Y \in TM$ and $k$ is a smooth function.

M.M.Tripathy and Jeong - Jik Kim [6] introduced the notion of $N(k)$-quasi Einstein manifold which is defined as follows: If the generator $U$ belongs to the $k$-nullity distribution $N(k)$, then a quasi Einstein manifold $(M^n, g)$ is called $N(k)$-quasi Einstein manifold. Motivated by the above definitions we give the following definition.

**Definition 1.** Let $(M^n, g)$ be a non flat Riemannian manifold. If the ricci tensor $S$ of $(M^n, g)$ is non zero and satisfies

$$S(X,Y) = a g(X,Y) + b A(X)B(Y) + c B(X)A(Y), \tag{1}$$

where $a,b$ and $c$ are smooth functions and $A$ and $B$ are non zero 1-forms such that $g(X,U) = A(X)$ and $g(X,V) = B(X)$ for all vector fields $X$, and $U$ and $V$ being the orthogonal unit vector fields called generators of the manifold belong to $N(k)$, then we say that $(M^n, g)$ is a $N(k)$-mixed quasi Einstein manifold and is denoted by $N(k)-(MQE)_n$.

In this paper we introduce another notion of a manifold of mixed quasi constant curvature similar to manifold of quasi constant curvature defined in [4]. A Riemannian manifold $(M^n, g)$ is called a manifold of mixed quasi constant curvature if it is conformally flat and the curvature tensor $'R$ of type (0,4) satisfies the condition

$$'R(X,Y,Z,W) = p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q [g(X,W)A(Y)B(Z) - g(X,Z)A(Y)B(W) + g(X,W)A(Z)B(Y) - g(X,Z)A(W)B(Y)] + s [g(Y,Z)A(W)B(X) - g(Y,W)A(Z)B(X) + g(Y,Z)A(X)B(W) - g(Y,W)A(X)B(Z)] \tag{2}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold. Taking $X = W = e_i$ and summing over $i, 1 \leq i \leq n$ in (2), we obtain

$$S(Y,Z) = (n-1)p g(Y,Z) + (n-1)q [A(Y)B(Z) + A(Z)B(Y)] + s [2g(Y,Z) - A(Z)B(Y) - A(Y)B(Z)]$$

which implies

$$S(Y,Z) = a g(Y,Z) + b A(Y)B(Z) + c A(Z)B(Y) \tag{3}$$

where $b = c = (n-1)q - s, a = (n-1)p + 2s$. i.e. the space $(M^n, g)$ is mixed quasi Einstein. Thus we have

**Theorem 1.** A manifold of mixed quasi constant curvature is a mixed quasi Einstein manifold.
Conversely suppose \((M^n, g)\) is conformally flat mixed quasi Einstein manifold. Then
\[
R(X, Y)Z = \frac{1}{n-2} \{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\}
- \frac{r}{(n-1)(n-2)} \{g(Y,Z)X - g(X,Z)Y\}. \tag{4}
\]

Here \(Q\) is Ricci operator defined by \(S(X, Y) = g(QX, Y)\).

From the above equation, we get
\[
R(X, Y, Z, W) = g(R(X, Y)Z, W)
= \frac{1}{n-2} \{g(Y,Z)S(X, W) - g(X,Z)S(Y, W)
+ S(Y,Z)g(X, W) - S(X,Z)g(Y, W)\}
- \frac{r}{(n-1)(n-2)} \{g(Y,Z)g(X, W) - g(X,Z)g(Y, W)\}. \tag{5}
\]

Taking \(X = Y = e_i\) and taking summation over \(i, 1 \leq i \leq n\) in (1), we obtain \(r = na\).
Substituting this in (5) and using (1), we get
\[
R(X, Y, Z, W) = p[g(Y,Z)g(X, W) - g(X,Z)g(Y, W)]
+ q[g(X, W)A(Y)B(Z) - g(X,Z)A(Y)B(W) + g(X, W)A(Z)B(Y)
- g(X, Z)A(W)B(Y)] + s[g(Y,Z)A(W)B(X) - g(Y, W)A(Z)B(X)
+ g(Y, Z)A(X)B(W) - g(Y, W)A(X)B(Z)]
\]
where \(p = \frac{a}{n-1}, q = \frac{b}{n-2}, s = \frac{c}{n-2}\), i.e. \((M^n, g)\) is a manifold of mixed quasi constant curvature.

2. Existence Theorem of a \(N(k)\)-mixed Quasi Einstein Manifolds

**Theorem 2.** If in a conformally flat Riemannian manifold \((M^n, g)\), the ricci tensor \(S\) satisfies the relation
\[
S(X, Z)g(Y, W) - S(Y, Z)g(X, W) = \beta(g(Y, Z)S(X, W) - g(X, Z)S(Y, W)) \tag{6}
\]
where \(\beta\) is a non zero scalar, then \((M^n, g)\) is a \(N(k)\)-mixed quasi Einstein manifold.

**Proof.** Let \(U\) be a vector field defined by \(g(X, U) = A(X)\), \(\forall X \in TM\).
Taking \(X = W = U\) in (6), we obtain
\[
S(Y, Z) = ag(Y, Z) + bA(Y)B(Z) + cA(Z)B(Y) \tag{7}
\]
where \(a = \frac{-\alpha\beta}{u}, \alpha = S(U, U), u = g(U, U), b = \frac{1}{u}, c = \frac{\beta}{u}\), and \(S(U, Z) = S(Z, U) = g(QZ, U) = A(QZ) = B(Z)\). Therefore \((M^n, g)\) is mixed quasi Einstein.
If \((M^n, g)\) is conformally flat, then taking \(Z = U\) in (4), we obtain

\[
R(X, Y)U = \frac{1}{n-2}[A(Y)QX - A(X)QY + S(Y, U)X - S(U, Y)X - S(Y, U)X - S(X, U)Y] 
- \frac{r}{(n-1)(n-2)}[A(Y)X - A(X)Y] 
\]

(8)

Therefore we have

\[
g \]

which can be rewritten as

\[
\]  

Taking \(\beta = 1\) in (6), we get

\[
S(X, Z)g(Y, W) - S(Y, Z)g(X, W) - g(Y, Z)S(X, W) + g(X, Z)S(Y, W) = 0 
\]

Taking \(Z = U\) in the above equation, we obtain

\[
S(X, U)g(Y, W) - S(Y, U)g(X, W) - A(Y)S(X, W) + A(X)S(Y, W) = 0, 
\]

which can be rewritten as \(g(S(X, U)Y - S(Y, U)X - A(Y)QX + A(X)QY, W) = 0, \forall W.\)

Therefore we have \(S(X, U)Y - S(Y, U)X - A(Y)QX + A(X)QY = 0.\)

Substituting this in (8), we get \(R(X, Y)U = k(A(Y)X - A(X)Y)\), where \(k = \frac{-r}{(n-1)(n-2)}.\)

Therefore we have \(U \in \mathbb{N}_p(k)\), where \(k = \frac{-r}{(n-1)(n-2)}.\)

Suppose \(V\) is a unit vector field orthogonal to \(U\). Then, we have \(V \in \mathbb{N}_p(k).\)

Hence \((M^n, g)\) is a \(N(k)\)-mixed quasi Einstein manifold.

As it is well known that a 3-dimensional Riemannian manifold is conformally flat.

Thus we have

**Corollary 1.** A 3-dimensional manifold is \(N(k)\)\(-\)mixed quasi Einstein manifold provided (6) holds.

### 3. Example of a \(N(k)\)\(-\)\((MQE)_n\) manifold

Let \((M^n, \tilde{g})\) be a hypersurface of the Euclidean space \(E^{n+1}\). Let \(A\) be a \((1,1)\) tensor corresponding to the normal valued second fundamental tensor \(H\).

\[
\tilde{g}(A_\xi(X), Y) = g(H(X, Y), \xi) 
\]

(10)

where \(\xi\) is a unit normal vector field and \(X\) and \(Y\) are tangent vector fields.

Further

\[
H_\xi(X, Y) = \tilde{g}(A_\xi(X), Y) 
\]

(11)

The hypersurface \((M^n, \tilde{g})\) is quasi umbilical if

\[
H_\xi(X, Y) = a\tilde{g}(X, Y) + \beta C(X)D(Y) 
\]

(12)

In view of (10), we have

\[
H(X, Y) = a\tilde{g}(X, Y)\xi + \beta C(X)D(Y)\xi. 
\]

(13)
The Gauss equation of $M^n$ in $E^{n+1}$ can be written as
\begin{equation}
\bar{g}(\bar{R}(X,Y)Z,W) = \bar{g}(H(X,W),H(Y,Z)) - \bar{g}(H(W,Y),H(Z,X)) \tag{14}
\end{equation}

From (12) and (14), we have

\begin{align*}
\bar{R}(X,Y,Z,W) &= \alpha^2 g(X,W)g(Y,Z) + \alpha\beta g(X,W)C(Y)D(Z) \\
&+ \alpha\beta g(Y,Z)C(X)D(W) + \beta^2 C(X)C(Y)D(W)D(Z) \\
&- \alpha^2 g(W,Y)g(Z,X) - \alpha\beta g(W,Y)C(Z)D(X) \\
&- \alpha\beta g(Z,X)C(W)D(Y) - \beta^2 C(W)D(Y)C(Z)D(Z)
\end{align*}

Contracting the above equation with $X = W = e_i$ and taking summation over $i, 1 \leq i \leq n$, we obtain

\begin{equation}
\bar{S}(Y,Z) = a g(Y,Z) + b C(Y)D(Z) + c C(Z)D(Y)
\end{equation}

where $a = (n-1)\alpha^2, b = (n-1)\alpha\beta + \beta^2, c = -\beta(2\alpha + \beta)$.

Hence $(M^n, \bar{g})$ is a mixed quasi Einstein manifold.

Suppose $U$ and $V$ are unit orthogonal vectorfields corresponding to the 1-forms $C$ and $D$ respectively. Then putting $Z = U$ in (13), we get

\begin{equation}
H(X,U) = \alpha C(X)\xi. \tag{15}
\end{equation}

Putting $Z = U$ in (14) and using (15), we get

\begin{equation}
\bar{R}(X,Y)U = k(C(Y)X - C(X)Y)
\end{equation}

where $k = \alpha^2$. Similarly we can show that

\begin{equation}
\bar{R}(X,Y)V = k(D(Y)X - D(X)Y)
\end{equation}

where $k = \alpha^2$. Thus we have

**Theorem 3.** A quasi umbilical hypersurface of a Euclidean space $E^{n+1}$ is a $N(k)$-mixed quasi Einstein manifold.

### 4. Ricci Curvature, Eigen Vectors and Associated Scalars of a $N(k)-(MQE)_n$

From (1) we have $S(U,U) = a = S(V,V), b = S(U,V) = S(V,U) = c$, since $g(U,V) = 0$.

Therefore only one of $b$ or $c$ is sufficient to define a mixed quasi Einstein space. A mixed quasi Einstein space may be defined as a Riemannian manifold in which ricci tensor $S$ satisfies

\begin{equation}
S(X,Y) = a g(X,Y) + b(A(X)B(Y) + B(X)A(Y)),
\end{equation}

It is well known that for a unit vector field $X$, $S(X,X)$ is the ricci curvature in the direction of $X$. Now if $X$ is a unit vector field in the section spanned by $U$ and $V$, then we have

\begin{equation}
1 = g(X,X) = g(aU + \beta V, aU + \beta V) = \alpha^2 + \beta^2,
\end{equation}
since \( g(U, V) = 0 \) and \( g(U, U) = g(V, V) = 1 \). Now
\[
S(X, X) = S(\alpha U + \beta V, \alpha U + \beta V) = a + 2bA(X)B(X).
\]

Thus we can state that

**Theorem 4.** In a \( N(k) - (MQE)_n \) manifold, the ricci curvature in the direction of both \( U \) and \( V \) is \( 'a' \) and the ricci curvature in all other directions of the section of \( U \) and \( V \) is \( a + 2bA(X)B(X) \).

Let \( (M^n, g) \) be a \( N(k) - (MQE)_n \) manifold.
Then \( S(U, U) = S(V, V) = a \) from which we get \( g(QU, U) = g(QV, V) = a \).
Since \( U, V \in N_p(k) \), we have,
\[
g(R(X, Y)U, W) = k \{A(Y)g(X, W) - A(X)g(Y, W)\}.
\]

Putting \( X = W = e_i \) and taking summation over \( i, 1 \leq i \leq n \), we obtain
\[
S(Y, U) = (n - 1)kA(X) \quad (16)
\]
Similarly we can get
\[
S(Y, V) = (n - 1)kB(X) \quad (17)
\]
From (1), we have
\[
S(X, U) = aA(X) + bB(X) \quad (18)
\]
\[
S(X, V) = bA(X) + aB(X) \quad (19)
\]
Substracting (17) from (16) and (19) from (18), and comparing the resulting equations, we obtain
\[
k = \frac{a - b}{n - 1}.
\]
Therefore
\[
S(X, U) = (a - b)g(X, U)
\]
and
\[
S(X, V) = (a - b)g(X, V).
\]
Therefore \( U \) and \( V \) are eigen vectors corresponding to the eigen value \( (a - b) \).

**5. Semi Symmetric and Ricci Symmetric \( N(k) - (MQE)_n \) Manifolds**

A Riemannian manifold \( (M^n, g) \) is semi symmetric if \( R(X, Y).R = 0, \forall X, Y \in TM \).
Since \( U \) and \( V \) are in \( N_p(k) \), we have
\[
R(X, Y)U = k(A(Y)X - A(X)Y) \quad (20)
\]
\[
R(X, Y)V = k(B(Y)X - B(X)Y) \quad (21)
\]
The equation (20) is equivalent to
\[ R(U, Y)Z = k \left( g(Y, Z)U - A(Z)Y \right) \] (22)
\[ R(X, U)Z = k \left( A(Z)X - g(X, Z)U \right) \] (23)

The equation (21) is equivalent to
\[ R(V, Y)Z = k \left( g(Y, Z)V - B(Z)Y \right) \] (24)
\[ R(X, V)Z = k \left( B(Z)X - g(X, Z)V \right) \] (25)

If \((M^n, g)\) is semi symmetric then we have

Putting \(X = U\) and \(T = V\) in (26), then using (21) and (22), we get
\[ k^2 \left\{ 2A(Z)B(Y)W + A(W)B(Z)Y - 2B(Z)g(Y, W)U \right\} = 0 \] (27)

From (27), we have
If \(k \neq 0\), then \(2A(Z)B(Y)W + A(W)B(Z)Y = 2B(Z)g(Y, W)U, \forall Y, Z, W \in TM\) holds.

Putting \(Z = V\) in the above equation, we get
\[ g(Y, W)U = A(W)Y \]

Taking covariant derivative on both sides of the above equation with respect to \(Z\), we obtain
\[ g(X, Y)\nabla_Z U = (ZA(Y)X - A(Y)) \nabla_Z X, \forall X, Y \]

Putting \(Y = V\), we get \(B(X)\nabla_Z U = 0\).
Since \(B(X) \neq 0\), we obtain \(\nabla_Z U = 0\).
i.e. \(U\) is a parallel vector field.

Similarly by taking \(X = V\) and \(T = U\) in (26), we obtain \(\nabla_Z V = 0\).
i.e. \(V\) is a parallel vector field.

Conversely suppose that \(U\) and \(V\) are parallel vector fields. Then \(\nabla_Z U = 0\) and \(\nabla_Z V = 0\), which then imply that
\[ R(X, Y)U = 0 \text{ and } R(X, Y)V = 0. \]
Substituting this in (26) with \(X = U\), we obtain \(R(U, X).R = 0\).
Similarly we get \(R(V, X).R = 0\).
Thus we can state that

**Theorem 5.** A \(N(k) - (MQE)_n\) manifold with \(k \neq 0\) satisfies \(R(U, X).R = 0 \text{ (or } R(V, X).R = 0)\) if and only if \(U \text{ (or } V\) is a parallel vector field.
Let \((M^n, g)\) be a \(N(k) - (MQE)_n\) ricci semi symmetric manifold. Then we have

\[ S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0 \]  

(28)

Putting \(X = V\) in (28) we obtain

\[ k \{ g(Y, Z)S(V, W) - B(Z)S(Y, W) + g(Y, W)S(Z, V) - B(W)S(Z, Y) \} = 0 \]

Putting \(W = V\) in the above equation, we get

\[ k \left[ S(Z, Y) - a g(Y, Z) + bA(Y)B(Z) - bA(Z)B(Y) \right] = 0 \]

If \(k \neq 0\) then we have \(S(Z, Y) = a g(Y, Z) - b A(Y) B(Z) + b A(Z) B(Y)\).

Comparing this with (1), we obtain \(b + c = 0\).

But we have \(b - c = 0\), \{ section 4 \}

Therefore \(b = 0\) and \(c = 0\). i.e. \((M^n, g)\) reduces to Einstein space which it is not.

Therefore we must have \(k = 0\).

Conversely suppose \(k = 0\). Then we obtain \(R(V, X)Y = 0\) which implies \(R(V, X).S = 0\). Similarly, we have, \(R(U, X).S = 0\). if and only if \(k = 0\).

Thus we have,

**Theorem 6.** A \(N(k) - (MQE)_n\) manifold satisfies \(R(V, X).S = 0\). and \(R(U, X).S = 0\) if and only if \(k = 0\).

6. Ricci Recurrent \(N(k) - (MQE)_n\) Manifolds

Let \((M^n, g)\) be a \(N(k) - (MQE)_n\) manifold. If \(U\) and \(V\) are parallel vector fields, then \(\nabla_X U = 0\) and \(\nabla_X V = 0\).

From which we get that \(R(X, Y)U = 0\) and \(R(X, Y)V = 0\). Therefore

\[ S(X, U) = 0, S(X, V) = 0 \]  

(29)

From (1), we have

\[ S(X, U) = aA(X) + bB(X) \text{and} \]

\[ S(X, V) = aB(X) + bA(X) \]  

(30)

(31)

From (29), (30) and (31), we have \(a = b\).

Therefore we can rewrite the equation (1) in the following form:

\[ S(X, Y) = a \{ g(X, Y) + A(X)B(Y) + B(X)A(Y) \} . \]

Taking the covariant derivative of the above equation with respect to \(Z\), we obtain

\[ \nabla_Z S(X, Y) = d a(Z) \{ g(X, Y) + A(X)B(Y) + B(X)A(Y) \} \]
since $\nabla_X U = 0$ and $\nabla_X V = 0$ imply that $\nabla_Z A(X) = 0$ and $\nabla_Z B(X) = 0.$ Therefore $(\nabla_Z S)(X,Y) = \frac{d a(Z)}{a} S(X,Y),$ i.e. the manifold $(M^n,g)$ is ricci recurrent.

Conversely, suppose that $N(k) - (MQE)_n$ manifold is ricci recurrent. Then

$$(\nabla_X S)(Y,Z) = D(X) S(Y,Z), D(X) \neq 0.$$ But

$$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y,Z) - S(Y,\nabla_X Z)$$

Therefore

$$D(X) S(Y,Z) = XS(Y,Z) - S(\nabla_X Y,Z) - S(Y,\nabla_X Z)$$

Putting $Y = Z = U,$ we obtain

$$X a - a D(X) = 2 a \left( g(\nabla_X U, U) + B(\nabla_X U) \right)$$
i.e. $(d a - a D) X = 2 a B(\nabla_X U).$ since $g(U, U) = 1$ implies $g(\nabla_X U, U) = 0.$ Therefore $B(\nabla_X U) = 0$ if and only if

$$(d a)(X) = a D(X)$$ (32)

But $B(\nabla_X U) = 0$ implies that either $U$ is a parallel vector field or $\nabla_X U \perp V.$ Similarly we have, if (32) holds then either $V$ is a parallel vector field or $\nabla_X V \perp U.$

Thus we can state that

**Theorem 7.** A $N(k)(MQE)_n$ manifold, where the generators $U$ and $V$ are parallel is a ricci recurrent manifold. Conversely suppose that $N(k) - (MQE)_n$ manifold is ricci recurrent, then either the vector field $U$ (or $V$) is parallel or $\nabla_X U \perp V$ (or $\nabla_X V \perp U$).

**References**


