QUINTUPLE PRODUCT IDENTITY AS A SPECIAL CASE OF RAMANUJAN'S $1\psi_1$ SUMMATION FORMULA

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In this note we observe an interesting fact that the well-known quintuple product identity can be regarded as a special case of the celebrated $1\psi_1$ summation formula of Ramanujan which is known to unify the Jacobi triple product identity and the q-binomial theorem.

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1. Introduction

It is well-known that the Jacobi’s triple product identity [2]

$$(-q^2;q^2)_{\infty} \left( \frac{-q}{z};q^2 \right)_{\infty} \left( q^2; q^2 \right)_{\infty} = \sum_{n=0}^{\infty} q^{n^2} z^n, \quad |q| < 1, \quad z \neq 0,$$  \hspace{1cm} (1.1)

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and the Euler-Cauchy q-binomial theorem

$$\frac{(at)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n}, \quad |q| < 1, \quad |t| < 1,$$

are special cases of the $_1\psi_1$ summation of Ramanujan [3, p. 196], [1, p. 32].

If $|\beta q| < |z| < 1$ and $|q| < 1$, then

$$1 + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{\beta n}; q^2\right)_n (-\alpha q z)^n}{\left(\frac{1}{\beta} q^2; q^2\right)_n} + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{\beta n}; q^2\right)_n (-\frac{\beta q}{n})^n}{\left(\frac{1}{\beta} q^2; q^2\right)_n}$$

$$= \frac{(-q z; q^2)_\infty}{(-\alpha q z; q^2)_\infty} \frac{(-\frac{\alpha q}{n}; q^2)_\infty}{(-\frac{\frac{\beta q}{n}}{q^2}; q^2)_\infty} (\alpha q^2; q^2)_\infty (\beta q^2; q^2)_\infty.$$

Here, as usual,

$$(a)_0 : = (a; q)_0 = 1,$$

$$(a)_\infty : = (a; q)_\infty : = \prod_{n=0}^{\infty} (1 - a q^n), \quad |q| < 1,$$

and

$$(a)_n : = \frac{(a)_\infty}{(aq^n)_\infty}, \quad n : \text{any integer}.$$

In fact, put $\alpha = \beta = 0$ in (1.3) to obtain (1.1) and put $\alpha = \frac{1}{n}, \beta = \frac{4}{n}, z = \frac{\alpha q}{n}$ and then replace $q^2$ by $q$ and $b$ by $q$ in (1.3) to obtain (1.2). The purpose of the present paper is to show that the quintuple product identity given by (2.1) below can also be regarded as a special case of the $_1\psi_1$ summation (1.3). It is however customary to write the quintuple product identity in the equivalent form

$$(-x)_\infty \left(\frac{-q}{x}\right)_\infty (q)_\infty \left(\frac{q x^2; q^2}{x}\right)_\infty \left(\frac{x^2 q^2}{x}\right)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2 + 3n} (x^{3n+1} + x^{-3n}).$$

This equivalence is evident on using the easily verifiable identities

$$\frac{(x^2)_{\infty}}{(x)_{\infty}} = (-x)_{\infty} \left(\frac{x^2 q^2}{q}\right)_{\infty}$$

and

$$\frac{\left(\frac{q}{x}\right)_\infty}{\left(\frac{q}{x}\right)_\infty} = \left(-x\right)_{\infty} \left(\frac{q}{x}\right)_{\infty} \left(\frac{x^2 q^2}{x}\right)_{\infty},$$

on the left of (2.1).
2. Main Result

Theorem 2.1. If $|q|<1$ and $x \neq 0$, then
\[
\left(\frac{x^2}{q}\right)_\infty \frac{(\varphi \eta)}{(\varphi \xi)}_\infty = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} (x^{3n+1} + x^{-3n}).
\] (2.1)

Proof. The $1\psi_1$-summation formula (1.3) can be written as
\[
\left(\frac{x^2}{q}\right)_\infty \frac{(\varphi \eta)}{(\varphi \xi)}_\infty = \sum_{n=0}^{\infty} \frac{(\varphi \eta)}{(\varphi \xi)}_n \frac{\alpha^n x^{2n}}{(\beta q)_n} + \sum_{n=0}^{\infty} \frac{(\varphi \eta)}{(\varphi \xi)}_n \frac{\beta^n}{(\alpha q)_{n+1}}
\]
which gives
\[
\left(\frac{x^2}{q}\right)_\infty \frac{(\varphi \eta)}{(\varphi \xi)}_\infty = \sum_{n=0}^{\infty} (-1)^n q^\frac{n(n+1)}{2} x^{2n} - q x^2 \sum_{n=0}^{\infty} \frac{(\varphi \eta)}{(\varphi \xi)}_n \frac{q^n}{x^n}.
\] (2.2)
Comparing this with (2.1), we now need only write the right side in power series of $x$. For this, put $b = q, a = \frac{1}{\alpha}, t = -\alpha q^{\frac{1}{2}} x$ and $c = \beta q$ in the well-known Heine's transformation [1]
\[
\sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(c)_n (q)_n} = \frac{\varphi \chi \eta}{(c)_\infty (q)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n (\frac{\alpha x}{\beta})}{(bt)_n (q)_n} (\frac{c}{q})^n
\]
to obtain
\[
\sum_{n=0}^{\infty} \frac{(\frac{1}{\alpha})_n}{(\beta)_{n+1}} (-\alpha q^{\frac{1}{2}} x)^n = \sum_{n=0}^{\infty} \frac{(-\alpha q^{\frac{1}{2}} x)^n}{(-\alpha q^{\frac{1}{2}} x)_{n+1}}.
\] (2.3)
Letting $\alpha$ to 0 in (2.3), replacing $x$ by $-q^{\frac{1}{2}} x^2$ and then putting $\beta = x$, we obtain
\[
\sum_{n=0}^{\infty} (-1)^n q^\frac{n(n+1)}{2} x^{2n} (x)_{n+1} = \sum_{n=0}^{\infty} (x)_n x^n.
\] (2.4)
Now,
\[
\sum_{n=0}^{\infty} (x)_n x^n = 1 + x(1 - x) \sum_{n=0}^{\infty} (xq)_n x^n
\] (2.5)
and
\[
(1 - x) \sum_{n=0}^{\infty} (xq)_n x^n = 1 - x^2 q \sum_{n=0}^{\infty} (xq)_n (xq)^n.
\] (2.6)
Employing (2.6) in (2.5), we obtain the functional equation
\[
\sum_{n=0}^{\infty} (x)_n x^n = 1 + x - x^3 q \sum_{n=0}^{\infty} (xq)_n (xq)^n.
\] (2.7)
Seeking the power series expansion
\[
\sum_{0}^{\infty} (x)_n x^n = \sum_{0}^{\infty} c_n(q) x^n,
\]
(2.7) gives
\[
\sum_{0}^{\infty} c_n(q) x^n = 1 + x - x^3 q \sum_{0}^{\infty} c_n(q) x^n q^n.
\]
Comparing the coefficients of \(x^n\) \((n \geq 1)\), we obtain
\[
c_{3n} = -q^{3n-2} c_{3n-3},
\]
\[
c_{3n+1} = -q^{3n-1} c_{3n-2}
\]
and
\[
c_{3n+2} = -q^{3} c_{3n-1}.
\]
These give on iteration,
\[
c_{3n} = (-1)^n q^{(3n-2)+(3n-5)+...+1} c_0(q) = (-1)^n q^{\frac{n(3n-1)}{2}},
\]
\[
c_{3n+1} = (-1)^n q^{\frac{n(3n+1)}{2}}
\]
and
\[
c_{3n+2} = 0.
\]
Thus we have the power series development,
\[
\sum_{0}^{\infty} (x)_n x^n = \sum_{0}^{\infty} c_n(q) x^n = \sum_{0}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} x^{3n} + \sum_{0}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} x^{3n+1}.
\]
(2.8)
Employing (2.8) in the second sum on the right of (2.2), we obtain the power series
\[
-\frac{q}{x^2} \sum_{0}^{\infty} \left(\frac{q}{x}\right)_n \left(\frac{q}{x}\right)^n = -\sum_{-\infty}^{1} (-1)^n q^{\frac{n(3n+1)}{2}} x^{3n+1} + \sum_{-\infty}^{1} (-1)^n q^{\frac{n(3n-1)}{2}} x^{3n}.
\]
(2.9)
Using (2.4), (2.8) and (2.9) in (2.2), we have the required identity (2.1).

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