QUINTUPLE PRODUCT IDENTITY AS A SPECIAL CASE OF RAMANUJAN'S ${}_1\psi_1$ SUMMATION FORMULA

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Communicated by K. Saito

Received March 17, 2010

Revised November 16, 2010

In this note we observe an interesting fact that the well-known quintuple product identity can be regarded as a special case of the celebrated ${}_1\psi_1$ summation formula of Ramanujan which is known to unify the Jacobi triple product identity and the q-binomial theorem.

Keywords: Ramanujan's ${}_1\psi_1$ summation; triple product identity; quintuple product identity.

AMS Subject Classification: 11S23, 33D20, 05A19

1. Introduction

It is well-known that the Jacobi's triple product identity [2]

$$(-qz; q^2)_\infty \left(\frac{-q}{z}; q^2 \right)_\infty (q^2; q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2} z^n, \quad |q| < 1, \quad z \neq 0, \quad (1.1)$$

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and the Euler-Cauchy q -binomial theorem

$$\frac{(at)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n}, \quad |q| < 1, \quad |t| < 1, \quad (1.2)$$

are special cases of the ${}_1\psi_1$ summation of Ramanujan [3, p. 196], [1, p. 32].

If $|\beta q| < |z| < \frac{1}{|\alpha q|}$ and $|q| < 1$, then

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{(\frac{1}{\alpha}; q^2)_n (-\alpha q z)^n}{(\beta q^2; q^2)_n} + \sum_{n=1}^{\infty} \frac{(\frac{1}{\beta}; q^2)_n \left(\frac{-\beta q}{z}\right)^n}{(\alpha q^2; q^2)_n} \\ = \frac{(-qz; q^2)_\infty \left(-\frac{q}{z}; q^2\right)_\infty (q^2; q^2)_\infty (\alpha \beta q^2; q^2)_\infty}{(-\alpha q z; q^2)_\infty \left(-\frac{\beta q}{z}; q^2\right)_\infty (\alpha q^2; q^2)_\infty (\beta q^2; q^2)_\infty}. \end{aligned} \quad (1.3)$$

Here, as usual,

$$(a)_0 := (a; q)_0 = 1,$$

$$(a)_\infty := (a; q)_\infty := \prod_{n=0}^{\infty} (1 - a q^n), \quad |q| < 1,$$

and

$$(a)_n := \frac{(a)_\infty}{(a q^n)_\infty}, \quad n : \text{any integer}.$$

In fact, put $\alpha = \beta = 0$ in (1.3) to obtain (1.1) and put $\alpha = \frac{1}{a}$, $\beta = \frac{b}{q^2}$, $z = \frac{-at}{q}$ and then replace q^2 by q and b by q in (1.3) to obtain (1.2). The purpose of the present paper is to show that the quintuple product identity given by (2.1) below can also be regarded as a special case of the ${}_1\psi_1$ summation (1.3). It is however customary to write the quintuple product identity in the equivalent form

$$(-x)_\infty \left(\frac{-q}{x}\right)_\infty (q)_\infty (x^2 q; q^2)_\infty \left(\frac{q}{x^2}; q^2\right)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}} (x^{3n+1} + x^{-3n}).$$

This equivalence is evident on using the easily verifiable identities

$$\frac{(x^2)_\infty}{(x)_\infty} = (-x)_\infty (x^2 q; q^2)_\infty$$

and

$$\frac{\left(\frac{q}{x^2}\right)_\infty}{\left(\frac{q}{x}\right)_\infty} = \left(\frac{-q}{x}\right)_\infty \left(\frac{q}{x^2}; q^2\right)_\infty,$$

on the left of (2.1).

2. Main Result

Theorem 2.1. *If $|q| < 1$ and $x \neq 0$, then*

$$\frac{(x^2)_\infty \left(\frac{q}{x^2}\right)_\infty (q)_\infty}{\left(\frac{q}{x}\right)_\infty (x)_\infty} = \sum_{-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}} (x^{3n+1} + x^{-3n}). \quad (2.1)$$

Proof. The ${}_1\psi_1$ -summation formula (1.3) can be written as

$$\frac{(x^2)_\infty \left(\frac{q}{x^2}\right)_\infty (q)_\infty (\alpha\beta q)_\infty}{(\alpha x^2)_\infty \left(\frac{\beta q}{x^2}\right)_\infty (\alpha q)_\infty (\beta q)_\infty} = \sum_0^{\infty} \frac{\left(\frac{1}{\alpha}\right)_n \alpha^n x^{2n}}{(\beta q)_n} + \sum_0^{\infty} \frac{\left(\frac{1}{\beta}\right)_{n+1} \left(\frac{\beta q}{x^2}\right)_{n+1}}{(\alpha q)_{n+1}}$$

which gives

$$\frac{(x^2)_\infty \left(\frac{q}{x^2}\right)_\infty (q)_\infty}{\left(\frac{q}{x}\right)_\infty (x)_\infty} = \sum_0^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} x^{2n}}{(x)_{n+1}} - \frac{q}{x^2} \sum_0^{\infty} \left(\frac{q}{x}\right)_n \left(\frac{q}{x}\right)^n. \quad (2.2)$$

Comparing this with (2.1), we now need only write the right side in power series of x . For this, put $b = q$, $a = \frac{1}{\alpha}$, $t = -\alpha q^{\frac{1}{2}} x$ and $c = \beta q$ in the well-known Heine's transformation [1]

$$\sum_0^{\infty} \frac{(a)_n (b)_n t^n}{(c)_n (q)_n} = \frac{\left(\frac{c}{b}\right)_\infty (bt)_\infty}{(c)_\infty (t)_\infty} \sum_0^{\infty} \frac{(b)_n \left(\frac{abt}{c}\right)_n}{(bt)_n (q)_n} \left(\frac{c}{q}\right)^n$$

to obtain

$$\sum_0^{\infty} \frac{\left(\frac{1}{\alpha}\right)_n \left(-\alpha q^{\frac{1}{2}} x\right)^n}{(\beta)_{n+1}} = \sum_0^{\infty} \frac{\left(\frac{-q^{\frac{1}{2}} x}{\beta}\right)_n \beta^n}{\left(-\alpha q^{\frac{1}{2}} x\right)_{n+1}}. \quad (2.3)$$

Letting α to 0 in (2.3), replacing x by $-q^{-\frac{1}{2}} x^2$ and then putting $\beta = x$, we obtain

$$\sum_0^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} x^{2n}}{(x)_{n+1}} = \sum_0^{\infty} (x)_n x^n. \quad (2.4)$$

Now,

$$\sum_0^{\infty} (x)_n x^n = 1 + x(1-x) \sum_0^{\infty} (xq)_n x^n \quad (2.5)$$

and

$$(1-x) \sum_0^{\infty} (xq)_n x^n = 1 - x^2 q \sum_0^{\infty} (xq)_n (xq)^n. \quad (2.6)$$

Employing (2.6) in (2.5), we obtain the functional equation

$$\sum_0^{\infty} (x)_n x^n = 1 + x - x^3 q \sum_0^{\infty} (xq)_n (xq)^n. \quad (2.7)$$

Seeking the power series expansion

$$\sum_0^{\infty} (x)_n x^n = \sum_0^{\infty} c_n(q) x^n,$$

(2.7) gives

$$\sum_0^{\infty} c_n(q) x^n = 1 + x - x^3 q \sum_0^{\infty} c_n(q) x^n q^n.$$

Comparing the coefficients of x^n ($n \geq 1$), we obtain

$$c_{3n} = -q^{3n-2} c_{3n-3},$$

$$c_{3n+1} = -q^{3n-1} c_{3n-2}$$

and

$$c_{3n+2} = -q^{3n} c_{3n-1}.$$

These give on iteration,

$$c_{3n} = (-1)^n q^{(3n-2)+(3n-5)+\dots+1} c_0(q) = (-1)^n q^{\frac{n(3n-1)}{2}},$$

$$c_{3n+1} = (-1)^n q^{\frac{n(3n+1)}{2}}$$

and

$$c_{3n+2} = 0.$$

Thus we have the power series development,

$$\sum_0^{\infty} (x)_n x^n = \sum_0^{\infty} c_n(q) x^n = \sum_0^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} x^{3n} + \sum_0^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} x^{3n+1}. \quad (2.8)$$

Employing (2.8) in the second sum on the right of (2.2), we obtain the power series

$$-\frac{q}{x^2} \sum_0^{\infty} \left(\frac{q}{x}\right)_n \left(\frac{q}{x}\right)^n = \sum_{-\infty}^{-1} (-1)^n q^{\frac{n(3n+1)}{2}} x^{3n+1} + \sum_{-\infty}^{-1} (-1)^n q^{\frac{n(3n-1)}{2}} x^{3n}. \quad (2.9)$$

Using (2.4), (2.8) and (2.9) in (2.2), we have the required identity (2.1). $\square$

Acknowledgment

The authors wish to thank the referees for their valuable suggestions.

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