On Weakly Symmetric and Weakly Conformally Symmetric Spaces admitting Veblen identities

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Abstract

In the present paper some properties involving curvature tensor, conformal curvature tensor, Ricci tensor and scalar curvature, on weakly symmetric, weakly conformally symmetric and pseudo symmetric spaces are obtained.

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1 Introduction

L. Tamassy and T.Q. Binh [3] have introduced the notion of weakly symmetric and weakly projective symmetric spaces. Based on this work, U.C. De and S. Bandyopadhyaa [7] introduced the notion of weakly conformally symmetric spaces and investigated some properties of such spaces. We consider these
spaces admitting Veblen identities and conformal Veblen identities [4] and determine some other properties.

Let $M_n$ be a Riemannian $n$-dimensional space covered by system of coordinate neighbourhoods $(U, x_i)$. Suppose $g_{ij}$, $R_{hijk}$ and $R_{ij}$ denote the local components of the metric tensor, the curvature tensor and the Ricci tensor, respectively, and let $R$ denote the scalar curvature. The non-flat Riemannian space $M_n$ $(n > 2)$ is called weakly symmetric space if the curvature tensor $R_{hijk}$ satisfies the condition [3]

\[ R_{hijk,l} = a_l R_{hijk} + b_h R_{tijk} + d_i R_{hljk} + e_j R_{hilk} + f_k R_{hij} \ (1.1) \]

where $a, b, d, e, f$ are 1-forms (non-zero simultaneously) and the ‘,’ denotes the covariant differentiation with respect to the metric tensor of the space. An $n$-dimensional weakly symmetric space $M_n$ is denoted by $(WS)_n$. Such spaces are studied by M.Pranovic [5], T.Q.Binh [6] and others. U.C.De and S.Bandyopadhyay [8] proved that the associated 1-forms $d$ and $f$ in (1.1) are identical with $b$ and $e$, respectively. Hence the condition (1.1) of $(WS)_n$ becomes

\[ R_{hijk,l} = a_l R_{hijk} + b_h R_{tijk} + b_i R_{hljk} + e_j R_{hilk} + e_k R_{hij} \ (1.2) \]

Further, the space $M_n$ is called pseudo symmetric if

\[ R_{hijk,l} = 2a_l R_{hijk} + a_h R_{tijk} + a_i R_{hljk} + a_j R_{hilk} + a_k R_{hij} \ . \]

An $n$-dimensional non conformally flat Riemannian space $M_n$ $(n > 3)$ is called weakly conformally symmetric if its conformal curvature tensor $C_{hijk}$, given by

\[ C_{hijk} = R_{hijk} + \frac{1}{n-2} (g_{hk} R_{ij} - g_{hj} R_{ik} + g_{ij} R_{hk} - g_{ik} R_{hj}) \]

\[ + \frac{R}{(n-1)(n-2)} (g_{hk} g_{ij} - g_{hj} g_{ik}) \]

satisfies the condition

\[ C_{hijk,l} = a_l C_{hijk} + b_h C_{tijk} + d_i C_{hljk} + e_j C_{hilk} + f_k C_{hij} \ (1.3) \]

where $a, b, d, e, f$ are associated 1-forms (non-zero simultaneously). A weakly conformally symmetric space $M_n$ is denoted by $(WCS)_n$. As in case of $(WS)_n$ it is proved that $d$ and $f$ are identical with $b$ and $e$ respectively. So, (1.4) reduces to

\[ C_{hijk,l} = a_l C_{hijk} + b_h C_{tijk} + b_i C_{hljk} + e_j C_{hilk} + e_k C_{hij} \]
for weakly conformally symmetric spaces. The space $M_n$ is called pseudo conformally symmetric if

\[ C_{hijk,l} = 2a_lC_{hijk} + a_kC_{lijk} + a_iC_{hilk} + a_jC_{hil} + a_lC_{hijl} . \]

The conformal curvature tensor satisfies the conditions:

\[ C_{hijk,l} + C_{jki,l} + C_{kij,l} = 0 , \tag{1.6} \]
\[ C_{rjk} = C_{rik} = C_{ijr} = 0 , \tag{1.7} \]

and

\[ C_{hijk} = -C_{hikj} = C_{ikhj} = C_{kjih} . \tag{1.8} \]

A space is said to be quasi conformally flat if

\[ C'_{hijk} = 0 , \tag{1.9} \]

where

\[ C'_{hijk} = aZ_{hijk} + b(ghkG_{ij} - gjkG_{ik} + gijG_{hk} - gikG_{hj}) , \tag{1.10} \]

with $a, b$ as arbitrary constants and

\[ Z_{hijk} = K_{hijk} - \frac{R}{n(n-1)} (ghkij - gijgh) , \quad G_{ij} = R_{ij} - \frac{R}{n} g_{ij}. \]

The Veblen identities and conformal Veblen identities in $M_n$ are given by [4]:

\[ V_{ijkl}^h = R_{ijkl}^h + R_{kijl}^h + R_{lijk}^h + R_{ijlk}^h = 0 \tag{1.11} \]

and

\[ W_{ijkl}^h = C_{ijkl}^h + C_{klji}^h + C_{jilk}^h + C_{ijlk}^h \]

\[ -\frac{1}{(n-3)} g^{hm} \left\{ (g_{jm}C_{kil,p}^p + g_{km}C_{jli,p}^p + g_{im}C_{lkj,p}^p + g_{lm}C_{ijk,p}^p) \right\} - (g_{ik}C_{mlj,p}^p + g_{ij}C_{mlk,p}^p + g_{kl}C_{mji,p}^p + g_{lj}C_{mil,p}^p) \] \]

If the Ricci tensor satisfies the condition

\[ R_{ij} = \frac{R}{n} g_{ij} , \tag{1.13} \]

then $M_n$ is called Einstein space.
2 Weakly symmetric space \((WS)_n\).

By using (1.2) in the Bianchi identity,

\[
R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0,
\]

we get

\[
\beta_i R_{hijk} + \beta_j R_{hikl} + \beta_k R_{hilj} = 0,
\]

where we have put

\[
\beta_i = a_i - 2e_i
\]

and used the relations

\[
R_{hijk} + R_{hjki} + R_{hkij} = 0
\]

and

\[
R_{hijk} = -R_{ihjk} = R_{iwhj}
\]

satisfied by the curvature tensor field. By transvecting (2.2) by \(g^{lh}\) and using (2.5), we get

\[
\beta_h R_{ijh} = \beta_k R_{ji} - \beta_j R_{ki}.
\]

Now transvection of (2.6) with \(g^{ij}\) gives

\[
\beta_h R^h_k = \frac{\beta_k R}{2}.
\]

If \(M_n\) is an Einstein space, then (2.7) reduces to

\[
(n - 2) \beta_k R = 0.
\]

Now write the Veblen identity (1.11) in the form

\[
R_{hijk,l} + R_{hkil,j} + R_{hlkj,i} + R_{hjli,k} = 0
\]

and use (1.2) to get

\[
\alpha_i R_{hikj} + \alpha_j R_{hkil} + \alpha_k R_{hilj} + \alpha_l R_{hijk} = 0,
\]

where we have put

\[
\alpha_i = a_i - (b_i + e_i),
\]
and used the results (2.4) and (2.5). Transvecting (2.10) by $g^{kh}$ and using (2.5), we get

\[(2.12) \quad \alpha_h R_{jiti}^h = \alpha_i R_{ij} - \alpha_l R_{lij}.\]

Now by transvecting (2.12) with $g^{il}$ and using that $M_n$ is an Einstein space, we get

\[(2.13) \quad (n - 2) \alpha_i R = 0.\]

In view of (2.8) and (2.13) we have

**Theorem 2.1** The scalar curvature of weakly symmetric Einstein Riemannian space $M_n$ is zero provided the 1-form $a$ in (1.2) is neither $2e$ nor $b + e$.

**Remark 2.2** As per G. Herglotz [1] the scalar curvature of Einstein space $M_n$ is constant. But that constant is necessarily zero if $M_n$ is weakly symmetric in which $a \neq 2e, b + e$ for the 1-forms $a, b, e$ used in (1.2).

Suppose $R \neq 0$ in $M_n$, then from (2.8) and (2.13) we see that $\beta_i = 0$ and $\alpha_i = 0$, which in turn indicate $a = 2e, a = b + e$ and hence $b = e$. Hence $M_n$ reduces to pseudosymmetric. So, we state the following:

**Theorem 2.3** A weakly symmetric Einstein space with non-zero scalar curvature is pseudo symmetric.

### 3 Weakly conformally symmetric space ($WCS)_n$.

By using (1.5) in (1.12), we get

\[
a_t C_{hijk} + b_h C_{hijk} + b_t C_{hljk} + e_j C_{hiik} + e_k C_{hijl} +
\]

\[
+ a_j C_{hkil} + b_h C_{jkil} + b_k C_{hjil} + e_t C_{hkij} + e_t C_{hki} +
\]

\[
+ a_i C_{hlkj} + b_h C_{ilkj} + b_t C_{hikj} + e_t C_{hiij} + e_j C_{hikl} +
\]

\[
+ a_i C_{hjli} + b_h C_{kjli} + b_j C_{hkl} + e_t C_{hjki} + e_i C_{hjlk} -
\]

\[
\left(\frac{a_p + b_p}{n - 3}\right) \{g_{hkl} C_{ijkl}^{^p} + g_{jkh} C_{ki}^{^p} + g_{khi} C_{jk}^{^p} + g_{kh} C_{jki}^{^p}\}.
\]
\[-\{g_{kl}C^p_{hji} + g_{lj}C^p_{hik} + g_{ji}C^p_{hkl} + g_{ik}C^p_{hlj}\} = 0.\]

In view of (1.6), (1.7) and (1.8) above result reduces to

\[\alpha_l C_{hijk} + \alpha_j C_{hkl} + \alpha_k C_{hjl} - \left(\frac{a_p + b_p}{n - 3}\right) \{\{g_{ih}C^p_{ijk} + g_{jh}C^p_{hik} + g_{ij}C^p_{hkl} + g_{ik}C^p_{hlj}\}\} = 0,\]

where \(\alpha_l = a_l - (b_l + e_l)\). Contracting (3.1) by \(g^{hi}\) and using (1.6), (1.7) and (1.8), we get

\[\lambda_p C^p_{lkj} = 0,\]

where \(\lambda_p = 2b_p + e_p\).

Transvecting (3.1) by \(\lambda_l\) and using (1.7), (1.8) and (3.2), we get

\[(\lambda_l \alpha_l) C_{hijk} = \left(\frac{a_l + b_l}{n - 3}\right) \{\lambda_h C^l_{ijk} + \lambda_k C^l_{hij} + \lambda_j C^l_{hkl}\}.\]

Transvecting (3.3) with \(\lambda^h\) and using (3.2), we get

\[(\lambda^h \lambda_h) \left(\frac{a_l + b_l}{n - 3}\right) C^l_{ijk} = 0\]

and hence

\[(a_l + b_l) C^l_{ijk} = 0.\]

Now the equation (3.3), in view of (3.4), reduces to

\[(\lambda^l \alpha_l) C_{hijk} = 0.\]

So we state the following:

**Theorem 3.1** A weakly conformally symmetric space \((WCS)_n\), \(n > 3\), is conformally flat if the 1-forms \(a, b, c\) satisfy the condition \(\lambda^l \alpha_l \neq 0\), where \(\lambda_l = 2b_l + e_l\) and \(\alpha_l = a_l - (b_l + e_l)\).

Suppose \(\lambda^l \alpha_l = 0\). Then \(C_{hijk} \neq 0\). Now, by using

**Proposition A:** A quasi-conformally flat space is either conformally flat or Einstein.

We state the following:

**Theorem 3.2** A quasi conformally flat weakly conformally symmetric space \((WCS)_n\), \(n > 3\) with \(\lambda^l \alpha_l = 0\) is Einstein space.
Weakly symmetric and weakly conformally symmetric spaces

References


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