Abstract. In this paper we study the Randers manifolds of constant curvature induced by generalized Sasakian space forms with different almost contact structures.

1. Introduction

A Finsler metric $L(x, y)$ on an $n$-dimensional manifold $M^n$ is called an $(\alpha, \beta)$-metric\cite{9} $L(\alpha, \beta)$ if $L$ is positively homogeneous function of $\alpha$ and $\beta$ of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M^n$. Randers metrics are simplest $(\alpha, \beta)$-metrics, which are expressed in the form $F = \alpha + \beta$ with $\|\beta\| < 1$. In Finsler geometry it is difficult to make classification of Finsler metrics of constant curvature, but for the case of special Finsler metrics like Randers metrics, a classification theorem was given by Yasuda and Shimada\cite{16} and Matsumoto and Shimada\cite{14}. But Bao and Robles\cite{8} found some errors in both the proofs and they gave a corrected version of Yasuda and Shimada theorem which characterises Randers spaces of constant curvature.

Keywords and phrases: Randers metric, constant flag curvature, $a$-Sasakian, $b$-Kenmotsu, co-symplectic, $\phi$-sectional curvature, generalized Sasakian space forms, constant negative curvature.

Bejancu and Farran [6] gave a classification theorem for a class of proper Randers manifolds of positive constant flag curvature. They have shown that a Sasakian space form induces a Randers space of positive constant curvature. Same authors in [5] proved that under certain natural conditions, Randers manifolds of positive constant flag curvature are diffeomorphic to odd dimensional spheres. Hasegawa et al. [11] established that there are other almost contact structures that induce Randers space of constant negative flag curvature and constant curvature \( K = 0 \). Motivated by the above results, in this paper we investigate a class of Randers manifolds induced by generalized Sasakian space forms with \( a \)-Sasakian, \( b \)-Kenmotsu and co-symplectic structures.

2. Preliminaries

2.1. Randers Metrics. A Finsler metric \( F \) on a manifold \( M \) is a function \( F : TM \to (0, \infty) \) such that

1. \( F \) is \( C^\infty \) on \( TM \setminus \{0\} \)
2. \( F_x = F|_{T_x M} \) is a Minkowski norm on \( T_x M \):
   \[ F(x, \lambda y) = \lambda F(x, y) \text{ for } \lambda > 0 \]
   The fundamental tensor field is positive definite, \( (g_{ij}(x, y)) > 0 \), where \( g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \).

Randers metrics are among the simplest Finsler metrics. Let \( F = \alpha + \beta \) be a Randers metric on a manifold \( M^n \), where \( \alpha^2 = a_{ij}(x)y^iy^j \) is a Riemannian metric and \( \beta = b_i(x)y^i \) is a 1-form on \( M^n \) with \( \|\beta\| < 1 \) [9]. The space \( R^n = (M^n, \alpha) \) is called the associated Riemannian space with \( F^n = (M^n, L(\alpha, \beta)) \). The spray coefficients \( G^i \) of \( F \) are given by

\[
G^j = \frac{1}{4} g^{jh} \left( \frac{\partial^2 F^2}{\partial y^h \partial x^k y^k} - \frac{\partial F^2}{\partial x^h} \right),
\]

The spray coefficients \( G^i \) of \( F \) and \( G^i \alpha \) of \( \alpha \) are related by [17]

\[
G^i = G^i_{\alpha} + Py^i + Q^i,
\]

where

\[
P = \frac{e^0_0}{2F} - s_0, Q^i = \alpha s^i_0,
\]

\[
e_{ij} = r_{ij} + b_is_j + b_js_i,
\]

\[
r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}),
\]

\[
s^i_j = a^h s_{hj}, s_j = b_is^i_j = b^i s_{ij},
\]

\[
e_{00} = e_{ij} y^i y^j, s_0 = s_i y^i, s^i_0 = s^i_j y^j,
\]
where \( J \) denote the covariant differentiation with respect to the Levi-Civita connection \( \gamma^j_{ik}(x) \) of \( R^n \).

The spray coefficients \( G^i_j \) are used to define Riemann tensor \( R^i_j \) as follows.

\[
R^i_j = l^h \left( \frac{\delta}{\delta x^j} \left( \frac{G^k_h}{F} \right) - \frac{\delta}{\delta x^h} \left( \frac{G^k_j}{F} \right) \right), \quad l^h = \frac{y^h}{F},
\]

where

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \frac{G^j_i}{y^j} \frac{\partial}{\partial y^j}.
\]

A Finsler metric \( F \) has constant flag curvature \( \lambda \) if

\[
R^i_k = \lambda F^2 h^{i_k},
\]

where \( h^{i_k} = h_{jk} g^{ij}, h_{ij} = (g_{ij} - l_l l_l), l_i = l^l g_{lj} \).

The Ricci curvature and Ricci scalar are defined by

\[
Ric = R^i_i, \quad R = \frac{1}{n+1} Ric.
\]

A Finsler metric \( F \) has constant Ricci curvature if

\[
Ric = (n-1)\lambda F^2.
\]

2.2. Almost contact metric structures and Generalized Sasakian space forms. An \( n \)-dimensional Riemannian manifold \((M, \alpha)\) is said to be an almost contact metric manifold if there exist on \( M \), a \((1,1)\) tensor field \( \phi \), a unit vector field \( \xi \), a 1-form \( \eta \) satisfying

\[
\varphi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \varphi \xi = 0, \eta \circ \varphi = 0.
\]

\[
a(\phi X, \phi Y) = a(X, Y) - \eta(X)\eta(Y),
\]

\[
a(X, \phi Y) + a(\phi X, Y) = 0, \forall X, Y \in TM.
\]

We denote an almost contact metric manifold by \( M(\phi, \xi, \eta, a) \). An almost contact metric manifold is called a contact metric manifold if \( d\eta = \Phi \), where \( \Phi \) is the fundamental 2-form on \( M \) given by \( \Phi(X, Y) = a(X, \phi Y) \). An almost contact metric manifold \( M(\phi, \xi, \eta, a) \) is called

- Sasakian manifold if

\[
(\nabla_X \phi)Y = a(X, Y)\xi - \eta(Y)X,
\]

- Kenmotsu manifold if

\[
(\nabla_X \phi)Y = -a(X, Y)\xi + \eta(Y)\phi X,
\]
• a (a,b)-trans-Sasakian manifold if
\[
(\nabla_X \phi) Y = a(a(X,Y)\xi - \eta(Y)X) + b(a(\phi X, Y)\xi - \eta(Y)\phi X),
\]
where \(a\) and \(b\) are some functions.

In a trans-Sasakian manifold \(M\), the following hold:
\[
(\nabla_X \xi) = -\phi X + \beta(X - \eta(X)\xi), \quad (\nabla_X \eta) Y = a(\nabla_X \xi, Y), \forall X \in TM.
\]
If \(b = 0\), \(M\) is said to be an \(a\)- Sasakian manifold and Sasakian mani-
folds are the examples of \(a\)- Sasakian manifolds with \(a = 1\).
If \(a = 0\),\(M\) is said to be an \(b\)- Kenmotsu manifold and Kenmotsu
manifolds are examples of \(b\)- Kenmotsu manifolds with \(b = 1\).
If \(X\) is orthonormal to \(\xi\) then the plane section \(\{X, \phi X\}\) is called
\(\phi\)-section and the curvature associated with this section is called \(\phi\)-
sectional curvature which is given by
\[
H(X) = K(X, \phi X) = R(X, \phi X, X, \phi X).
\]
A Sasakian manifold of constant \(\phi\)-sectional curvature \(c\) is called a
Sasakian space form \(M(c)\).
A Kenmotsu manifold of constant \(\phi\)-sectional curvature \(c\) is called a
Kenmotsu space form.
An almost contact metric manifold \((M, \phi, \xi, \eta, a)\) is called a generalized
Sasakian space forms if the curvature tensor \(R\) is given by
\[
R(X, Y) Z = f_1\{g(Y, Z)X - g(X, Z)Y\} \\
+ f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\
+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\},
\]
where \(f_1, f_2, f_3\) differential functions on \(M\).
Sasakian space forms are examples of generalized Sasakian space forms
with constant functions \(f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}\).

3. Randers spaces of constant curvature induced by
generalized Sasakian space forms

Yasuda-Shimada[16] gave the following calssical theorem for Ran-
ders spaces of constant curvature.

\textbf{Theorem 1.} Let \(F^n = (M, F = \alpha + \beta), \alpha = \sqrt{a_{ij}(x)y^iy^j}, \beta = b_i(x)y^i\)
be a Randers space. Then
\[ [A] \ F^n \ has \ constant \ negative \ flag \ curvature \ \lambda \ and \ \curl b_i = b^j(b_{j[i} - b_{i]}), = 0 \ if \ and \ only \ if \ the \ Riemannian \ space \ (M, \alpha) \ is \ of \ negative \]
constant sectional curvature $-\lambda^2$. $b_{ij} = \lambda (a_{ij} - b_i b_j)$,
[B] $F^n$ is flat (i.e. $\lambda = 0$) and $\text{curl} b_i = 0$ if and only if it is locally Minkowski,
[C] $F^n$ has constant positive flag curvature $\lambda$ and $\text{curl} b_i = 0$ if and only if the Riemannian curvature tensor $R$ of the Riemannian space $(M, \alpha)$ satisfies
\[
R_{hikj} = \lambda(b_{hk} b_j a_{ik} - b_k b_h a_{ij}) + \lambda(b_i b_k a_{hj} - b_i b_j a_{kh})
+ \lambda(1 - \|b\|^2)(a_{hj} a_{ik} - a_{kh} a_{ij})
+ 2 b_{ij} b_{jk} - b_{ik} b_{hj} - b_{ij} b_{kj},
\]
and $\|b\|$ is a constant and $b_{j|k} + b_{k|j} = 0$.

In an $\alpha$-Sasakian manifold,
\[
\eta_{i|j} = -a \phi_i^j a_{jk},
\eta_{i|j} + \eta_{j|i} = 0,
\eta_{i|j} - \eta_{j|i} = 2 a \phi_i^j a_{jk},
\eta_{i|j} \xi^i = \eta_{j|i} \xi^j = 0,
\xi^i (\eta_{i|j} - \eta_{j|i}) = 0.
\]
The expression for $R$ in a generalized Sasakian space form is
\[
R_{hijk} = f_1 (a_{hj} a_{ik} - a_{hk} a_{ij}) - \frac{f_2}{a^2} (\eta_{i|j} \eta_{h|k} - \eta_{h|i} \eta_{j|k} + 2 \eta_{i|h} \eta_{j|k})
+ f_3 (\eta_{i|j} a_{hk} - \eta_{h} a_{jk} + \eta_{k} a_{ij} - \eta_{i} a_{hj}).
\]
Suppose $M$ is an odd dimensional almost contact metric manifold with almost contact metric structure $(\phi, \xi, \eta, a)$. Then we have a Riemannian metric $a$ and a 1-form $\eta = \eta_i(x) dx^i$ on $M$. We can construct in a natural way a Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x) y^j y^i}$, $\beta = c \eta_i(x) y^i$, with $0 < c < 1$. Then $F$ is a Randers metric on $M$ which is positive definite. This is because $\|b\|^2 = c^2 \|\eta\|^2 = c^2 < 1$. Further we have
\[
(1) \quad \|b\| = c, \text{ a constant.}
(2) \quad (\nabla_{\phi Y} b) Y = c (\nabla_{\phi Y} \eta) Y = -ac (\phi Y, \phi Y) \neq 0. \text{ i.e. } b \text{ is not parallel.}
(3) \quad b_{i|j} + b_{j|i} = c (\eta_{i|j} + \eta_{j|i}) = 0, \text{ follows from 19.}
\]

**Theorem 2** (P. Alegre and A. Carriazo, 2008). If $M(f_1, f_2, f_3)$ is an $\alpha$-Sasakian generalized Sasakian space form, then $\xi(a) = 0$ and $f_1 - f_3 = a^2$ holds. Further if $M$ is connected then $a$ is a constant.
If we choose $f_2 = f_3 = -c^2a^2$, then from the above theorem, $f_1 = -a^2(1 - c^2)$. The equation (23) takes the form
\[ R_{hijk} = \lambda(1 - \|b\|^2)(a_{hj}a_{ik} - a_{hk}a_{ij}) \]
(24) \[ + \lambda(2\eta_{l[h}\eta_{j]k} - b_{ij}b_{k|h} - b_{h[j}b_{i]k}) \]
\[ + \lambda(b_hb_{hj} + b_jb_{h}a_{ik} - b_jb_{h}a_{hk} - b_hb_{k}a_{ij}), \text{with} \lambda = a^2. \]

Thus by theorem 1, we conclude that

**Theorem 3.** Let $(M, F = \alpha + \beta)$ be the Randers space induced by a connected $\alpha$-Sasakian generalized Sasakian space form $M(f_1, f_2, f_3)$, where $\alpha = \sqrt{\eta_{ij}(x)y^iy^j}$, $\beta = \eta_i(x)y^i$. If $f_2 = f_3 = -c^2a^2$ then the Randers space $(M, F)$ is a Randers space of positive constant curvature $\lambda = a^2$.

This theorem holds for $c = 1$ also. But the Randers metric is not positive definite. Randers-space of constant curvature 1. We define on $M$,

\[ a_{ij}^* = a^2a_{ij} \]
\[ b_i^*(x) = ab_i(x) \]
(25)

Then the function $F^*(x, y) = aF(x, y)$ is a Randers metric on $TM^0$. Now from $g_{ij} = \frac{1}{2} \frac{\partial F^2}{\partial y^i \partial y^j}$, we get

\[ g_{ij}^* = a^2g_{ij}, g^{ij*} = \frac{1}{a^2}g^{ij}, i^*_i = al_i, l^{*_i} = \frac{1}{a}l^i, \]
\[ h_{ij}^* = a^2h_{ij}, h^{ij*} = \frac{1}{a^2}h^{ij}, h^{*_i}_j = h^*_i, \]
(26)

From (1), (4) and (6), it follows that the spray coefficients $G^{*_i}_j$ and the Riemann tensor $R^{*_i}_j$ are given by

\[ G^{*_i}_j = G^i, R^{*_i}_j = R^i_j. \]
(27)

If $F$ is a Randers metric of constant curvature $a^2$, then $R^i_j = a^2F^2h^i_j$. Therefore $R^{*_i}_j = R^i_j = a^2F^2h^i_j = F^{*_2}h^{*_i}_j$. This shows that the Randers metric $F^*$ is of constant curvature 1. This Randers metric is positive definite also. Thus we can state

**Theorem 4.** Let $(M, F = \alpha + \beta)$ be the Randers space induced by a connected $\alpha$-Sasakian generalized Sasakian space form $M(f_1, f_2, f_3)$, where $\alpha = \sqrt{\eta_{ij}(x)y^iy^j}$, $\beta = \eta_i(x)y^i$. If $f_2 = f_3 = -c^2a^2$ then the Randers space $(M, F^* = aF)$ is a Randers space of curvature 1.
In a $b$-Kenmotsu manifold, we have

$$
\eta_{ij} = b(a_{ij} - \eta_i \eta_j),
$$
(28)

$$
\eta_{ij} + \eta_{ji} = b(a_{ij} - \eta_i \eta_j),
$$
(29)

$$
\eta_{ij} \xi^i = \eta_{i} \xi^j = 0,
$$
(30)

$$
\xi^i (\eta_{ij} - \eta_{ji}) = 0.
$$
(31)

Suppose the Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}, \beta = \eta_i(x)y^i$ is induced by a $b$-Kenmotsu manifold. Then $F$ is a Randers metric on $M$ which is not positive definite. For a Randers metric $F = \alpha + \beta$ induced by a $b$-Kenmotsu manifold, we have $b_i = \eta_i$, and so $s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}) = 0, s_i = b^i s_{ij} = 0, r_{ij} = b(a_{ij} - b_i b_j)$ and $e_{00} = 2c(a_{ij} - b_i b_j)$, where $c = \frac{b}{2}$.

Hence the spray coefficients $G^i$ and the Riemann tensor $R^i_{\ j}$ of $F$ are given by

$$
G^i = G^i + \frac{\chi}{2F},
$$
(33)

$$
R^i_{\ k} = \alpha R^i_{\ k} + 3(\frac{\chi^2}{4F^2} - \frac{\Psi}{2F}) \left( \delta^i_{k} - \frac{F y^k y^i}{F} \right) + \tau_k y^i,
$$
(34)

where $\chi = b_{i|j} y^i y^j, \Psi = b_{i|j} y^i y^j y^k, \tau_k = \frac{1}{F}(b_{i|j} y^k y^i y^j) - \frac{1}{F} b_j \alpha R^i_{\ j}$. Suppose $F$ has constant Ricci curvature $\lambda$ and $\alpha$ has constant curvature $\mu$. Then

$$
Ric = (n - 1)\lambda F^2,
$$
(35)

$$
\alpha R^i_{\ k} = \mu \alpha^2 \left( \delta^i_{k} - \frac{\alpha y^k y^i}{\alpha} \right), \alpha Ric = (n - 1)\mu \alpha^2.
$$

From (34), we obtain

$$
Ric = \alpha Ric + 3(n - 1) \left( \left( \frac{\chi}{2F} \right)^2 - \frac{\Psi}{2F} \right)
$$
(36)

and $R^i_{\ k} = \lambda F^2 \delta^i_{k} + \tau_k y^i$, where $\tau_k y^k = -\lambda F^2$.

i.e. $F$ has constant curvature $\lambda$.

From (36), it follows that $F$ has constant Ricci curvature $\lambda$ if and only if

$$
\mu \alpha^2 + 3 \left( \left( \frac{\chi}{2F} \right)^2 - \frac{\Psi}{2F} \right) = \lambda F^2.
$$
(37)

The above equation is equivalent to following two equations:

$$
3\chi^2 = 2\beta \Psi + 4(\lambda - \mu)\alpha^4 + 4(6\lambda - \mu)\alpha^2 \beta^2 + 4\lambda \beta^2.
$$
(38)

$$
\Psi = 4(\mu - 2\lambda)\alpha^2 \beta - 8\lambda \beta^2.
$$
(39)
The equation (39) in (38) gives
\[ 3\chi^2 = 4(\lambda - \mu)\alpha^4 + 4(2\lambda + \mu)\alpha^2\beta^2 - 12\lambda\beta^4. \]
Differentiating covariantly and transvecting the resulting by \( y^k \), we obtain
\[ 3\Psi = 4(\mu + 2\lambda)\alpha^2\beta - 24\lambda\beta^3. \]
From (39) and (41), we obtain \( \mu = 4\lambda \). Putting this in (40), we obtain \( \chi^2 = -4\lambda(\alpha^2 - \beta^2) \).
This shows that \( \lambda \) is negative. Thus we have

**Theorem 5.** Let \((M, F = \alpha + \beta)\) be the Randers space induced by an \( b \)-Kenmotsu manifold, where \( \alpha = \sqrt{a_{ij}(x)y^iy^j}, \beta = \eta_i(x)y^i \). If \( F \) has constant Ricci curvature \( \lambda \) and \( \alpha \) has constant curvature \( \mu \), then \( \lambda = \frac{1}{4}\mu \) and this \( \lambda \) is negative.

We next see when \( \alpha \) has constant curvature. We have,

1. the \( \phi \)-sectional curvature of a generalized Sasakian space form \( M(f_1, f_2, f_3) \) is \( f_1 + 3f_2 \) [P. Alegre and A. Carriazo, 2008].
2. a Kenmotsu manifold of constant \( \phi \)-sectional curvature is a Riemannian space form of constant sectional curvature \(-1\) [Kenmotsu 1972].

Thus if \( f_1 \) and \( f_2 \) are constants, then \( b \)-Kenmotsu generalized Sasakian space form is a Riemannian space form of constant sectional curvature \(-1\). i.e. \( \alpha \) has constant curvature \(-1\). From the above discussions, we have

**Theorem 6.** Let \((M, F = \alpha + \beta)\) be the Randers space induced by an \( b \)-Kenmotsu generalized Sasakian space form with \( f_1, f_2 \) as constants. If \( F \) has constant Ricci curvature \( \lambda \), then \( \alpha \) has constant curvature \(-1 \) and \( F \) has constant curvature \( \lambda = -\frac{1}{4} \).

Let \( F = \alpha + \beta \) be a Randers metric induced by \( b \)-Kenmotsu manifold.
Since \( b_i = \eta_i \), from (29) and (30), we have
\[ s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}) = 0 \text{ and } e_{00} = 2c(a_{ij} - b_ib_j), \] where \( c = \frac{b}{2} \). We have the following theorem due to [Zhongmin Shen, 2003]:

**Theorem 7.** Let \( F = \alpha + \beta \) be a Randers metric of constant curvature \( K = \lambda \) on a manifold \( M \). Suppose that \( F \) satisfies \( e_{00} = 2c(\alpha^2 - \beta^2) \) for
some scalar function $c(x)$ on $M$ and $\beta$ is closed. Then $\lambda = -c^2 \leq 0$. $F$ is either locally Minkowskian $\lambda = 0$ or curvature in the form $\lambda = -c^2$.

Thus it follows that if $b = 0$, then $\lambda = 0$ and the Randers space $(M, F)$ reduces to a locally Minkowskian space. If $b \neq 0$, then the Randers space $(M, F)$ is of constant negative curvature $\lambda = -\frac{b^2}{4}$. Thus we state that

**Theorem 8.** If $M$ is an $b$-Kenmotsu generalized Sasakian space form with $f_1$ and $f_2$ are constants and if the induced Randers metric $F = \alpha + \beta$ has constant Ricci curvature, then $b$ must be equal to one.

**Acknowledgements:** This work is supported by CSIR 25(0179) / 10 / EMR - II.

**References**


