ON MEROMORPHIC FUNCTIONS THAT SHARE A SMALL FUNCTION WITH ITS DERIVATIVES

HARINA P. WAGHAMORE AND RAJESHWARI S.

(COMMUNICATE BY DAVID KALAJ)

Abstract. In this paper, we study the problem of meromorphic functions sharing a small function with its derivative and prove one theorem. The theorem improves the results of Jin-Dong Li and Guang-Xin Huang [10].

1. Introduction

Let $f$ be a nonconstant meromorphic function defined in the whole complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f), N(r,f)$ and so on, that can be found, for instance in [1].

Let $f$ and $g$ be two nonconstant meromorphic functions. Let $a$ be a finite complex number. We say that $f$ and $g$ share the value $a$ CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicites and we say that $f$ and $g$ share the value $a$ IM (ignoring multiplicities) if we do not consider the multiplicities. When $f$ and $g$ share $1$ IM, let $z_0$ be a 1-points of $f$ of order $p$, a 1-points of $g$ of order $q$, we denote by $N_{11}(r,\frac{1}{f-1})$ the counting function of those 1-points of $f$ and $g$ where $p = q = 1$; and $N_{E}^{(2)}(r,\frac{1}{f-1})$ the counting function of those 1-points of $f$ and $g$ where $p = q \geq 2$. $\overline{N}_L(r,\frac{1}{f-1})$ is the counting function of those 1-points of both $f$ and $g$ where $p > q$. In the same way, we can define $N_{11}(r,\frac{1}{g-1}), N_{E}^{(2)}(r,\frac{1}{g-1})$ and $\overline{N}_L(r,\frac{1}{g-1})$. If $f$ and $g$ share $1$ IM, it is easy to see that

$$\overline{N}(r,\frac{1}{f-1}) = N_{11}(r,\frac{1}{f-1}) + \overline{N}_L(r,\frac{1}{f-1}) + \overline{N}_L(r,\frac{1}{g-1}) + N_{E}^{(2)}(r,\frac{1}{g-1})$$

Let $f$ be a nonconstant meromorphic function. Let $a$ be a finite complex number, and $k$ be a positive integer, we denote by $N_k(r,\frac{1}{f-a}) (or \overline{N}_k(r,\frac{1}{f-a}))$ the counting function for zeros of $f - a$ with multiplicity $\leq k$ (ignoring multiplicities), and by $\overline{N}_k(r,\frac{1}{f-a}) (or \overline{N}_k(r,\frac{1}{f-a}))$ the counting function for zeros of $f - a$ with multiplicity

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atleast \( k \) (ignoring multiplicities). Set
\[
N_k(r, \frac{1}{f-a}) = N(r, \frac{1}{f-a}) + N(2r, \frac{1}{f-a}) + \cdots + N(kr, \frac{1}{f-a})
\]
\[
\Theta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}, \quad \delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.
\]
We further define
\[
\delta_k(a, f) = 1 - \limsup_{r \to \infty} \frac{N_k(r, \frac{1}{f-a})}{T(r, f)}.
\]
Clearly
\[
0 \leq \delta(a, f) \leq \delta_k(a, f) \leq \delta_{k-1}(a, f) \leq \cdots \leq \delta_2(a, f) \leq \delta_1(a, f) = \Theta(a, f)
\]

**Definition 1.1 (see [3]).** Let \( k \) be a nonnegative integer or infinity. For \( a \in \overline{\mathbb{C}} \) we denote by \( E_k(a, f) \) the set of all \( a \)-points of \( f \), where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k+1 \) times if \( m > k \). If \( E_k(a, f) = E_k(a, g) \), we say that \( f, g \) share the value \( a \) with weight \( k \).

We write \( f, g \) share \( (a, k) \) to mean that \( f, g \) share the value \( a \) with weight \( k \); clearly if \( f, g \) share \( (a, k) \), then \( f, g \) share \( (a, p) \) for all integers \( p \) with \( 0 \leq p \leq k \). Also, we note that \( f, g \) share a value \( a \) IM or CM if and only if they share \( (a, 0) \) or \( (a, \infty) \), respectively.

A meromorphic function \( a \) is said to be a small function of \( f \) where \( T(r, a) = S(r, f) \), that is \( T(r, a) = o(T(r, f)) \) as \( r \to \infty \), outside of a possible exceptional set of finite linear measure. Similarly, we can define that \( f \) and \( g \) share a small function \( a \) IM or CM or with weight \( k \).

R. Bruck [4] first considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

**Theorem A.** Let \( f \) be a non-constant entire function satisfying \( N(r, \frac{1}{f}) = S(r, f) \).

If \( f \) and \( f' \) share the value \( 1 \) CM, then \( \frac{f'-1}{f-1} = c \) for some nonzero constant \( c \).

Bruck [4] further posed the following conjecture.

**Conjecture 1.1.** Let \( f \) be a non-constant entire function, \( \rho_1(f) \) be the first iterated order of \( f \). If \( \rho_1(f) \) is not a positive integer or infinite, \( f \) and \( f' \) share the value \( 1 \) CM, then \( \frac{f'-1}{f-1} = c \) for some nonzero constant \( c \).

Yang [5] proved that the conjecture is true if \( f \) is an entire function of finite order. Y. C. Yang [6] considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

**Theorem B.** Let \( f \) be a non-constant entire function and \( a \equiv a(z)(\neq 0, \infty) \) be a meromorphic small function. If \( f - a \) and \( f(k) - a \) share \( 0 \) CM and \( \delta(0, f) > \frac{3}{4} \), then \( f \equiv f(k) \).

**Theorem C.** Let \( f \) be a non-constant non-entire meromorphic function and \( a \equiv a(z)(\neq 0, \infty) \) be a meromorphic small function. If
(i) \( f \) and \( a \) have no common poles.
(ii) \( f - a \) and \( f(k) - a \) share \( 0 \) CM.
(iii) \( 4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k \),
then \( f \equiv f(k) \) where \( k \) is a positive integer.

In the same paper, Yu [6] posed the following open questions.
(i) Can a CM share be replaced by an IM share value?
(ii) Can the condition \( \delta(0, f) > \frac{3}{4} \) of theorem B be further relaxed?
(iii) Can the condition (iii) in theorem C be further relaxed?
Can in general the condition (i) of theorem C be dropped?


**Theorem D.** Let \( f \) be a non-constant entire function and \( a \equiv a(z)(\neq 0, \infty) \) be a meromorphic small function. If \( f - a \) and \( f^{(k)} - a \) share 0 CM and \( \delta(0, f) > \frac{1}{2} \), then \( f \equiv f^{(k)} \).

Lahiri and Sarkar [8] gave some affirmative answers to the first three questions imposing some restrictions on the zeros and poles of \( a \). They obtained the following results.

**Theorem E.** Let \( f \) be a non-constant meromorphic function, \( k \) be a positive integer, and \( a \equiv a(z)(\neq 0, \infty) \) be a meromorphic small function. If

(i) \( a \) has no zero (pole) which is also a zero (pole) of \( f \) or \( f^{(k)} \) with the same multiplicity.

(ii) \( f - a \) and \( f^{(k)} - a \) share \((0, 2)\)

(iii) \( 2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k \) then \( f \equiv f^{(k)} \).

In 2005, Zhang [?] improved the above results and proved the following theorem.

**Theorem F.** Let \( f \) be a non-constant meromorphic function, \( k \geq 1 \), \( l \geq 0 \) be integers. Also let \( a \equiv a(z)(\neq 0, \infty) \) be a meromorphic small function. Suppose that \( f - a \) and \( f^{(k)} - a \) share \((0, l)\). If

\[ l \geq 2 \quad \text{and} \quad (3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4 \]  

or \( l = 1 \) and

\[ (4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6 \]  

or \( l = 0 \) and

\[ (6 + 2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10 \]

then \( f \equiv f^{(k)} \).

In 2015, Jin-Dong Li and Guang-Xiu Huang [?] proved the following Theorem.

**Theorem G.** Let \( f \) be a non-constant meromorphic function, \( k \geq 1 \), \( l \geq 0 \) be integers. Also let \( a \equiv a(z)(\neq 0, \infty) \) be a meromorphic small function. Suppose that \( f - a \) and \( f^{(k)} - a \) share \((0, l)\). If

\[ l \geq 2 \quad \text{and} \quad (3 + k)\Theta(\infty, f) + \delta_{2}(0, f) + \delta_{2+k}(0, f) > k + 4 \]  

\[ l = 1 \quad \text{and} \quad \left(\frac{7}{2} + k\right)\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \delta_{2}(0, f) + \delta_{2+k}(0, f) > k + 5 \]  

or \( l = 0 \) and

\[ (6 + 2k)\Theta(\infty, f) + 2\Theta(\infty, f) + \delta_{2}(0, f) + \delta_{1+k}(0, f) + \delta_{2+k}(0, f) > 2k + 10 \]

then \( f \equiv f^{(k)} \).

In this paper we pay our attention to the uniqueness of more generalised form of a function namely \( f^m \) and \( (f^n)^{(k)} \) sharing a small function for two arbitrary positive integer \( n \) and \( m \).

**Theorem 1.1.** Let \( f \) be a non-constant meromorphic function, \( k \geq 1 \), \( l \geq 0 \) be integers. Also let \( a \equiv a(z)(\neq 0, \infty) \) be a meromorphic small function. Suppose
that $f^m - a$ and $(f^n)^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and
$$ (k + 4)\Theta(\infty, f) + (k + 5)\Theta(0, f) > 2k + 9 - m \quad (1.7) $$

or $l = 1$ and
$$ (k + \frac{9}{2})\Theta(\infty, f) + (k + \frac{11}{2})\Theta(0, f) > 2k + 10 - m \quad (1.8) $$
or $l = 0$ and
$$ (2k + 7)\Theta(\infty, f) + (2k + 8)\Theta(0, f) > 4k + 15 - m \quad (1.9) $$
then $f^m \equiv (f^n)^{(k)}$.

Corollary 1.2. Let $f$ be a non-constant meromorphic function, $m, k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\neq 0, \infty)$ be a meromorphic small function. Suppose that $f^m - a$ and $(f^n)^{(k)} - a$ share $(0, l)$.

If $l \geq 2$ and $\Theta(0, f) > \frac{4}{5}$
or $l = 1$ and $\Theta(0, f) > \frac{9}{11}$
or $l = 0$ and $\Theta(0, f) > \frac{7}{8} - \frac{1}{8}[7\Theta(\infty, f) - 7\Theta(0, f)]$
then $f^m \equiv (f^n)^{(k)}$.

2. Lemmas

Lemma 2.1 (see [10]). Let $f$ be a non-constant meromorphic function, $k, p$ be two positive integers, then
$$ N_p(r, \frac{1}{f(k)}) \leq N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f) $$
clearly $\overline{N}(r, \frac{1}{f(k)}) = N_1(r, \frac{1}{f})$

Lemma 2.2 (see [10]). Let
$$ H = (\frac{F''}{F'} - \frac{2F'}{F - 1}) - (\frac{G''}{G'} - \frac{2G'}{G - 1}) \quad (2.1) $$
where $F$ and $G$ are two non constant meromorphic functions. If $F$ and $G$ share 1 IM and $H \neq 0$, then
$$ N_{11}(r, \frac{1}{F - 1}) \leq N(r, H) + S(r, F) + S(r, G) $$

Lemma 2.3 (see [11]). Let $f$ be a non-constant meromorphic function and let
$$ R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j} $$
be an irreducible rational function in $f$ with constant coefficients $a_k$ and $b_j$ where $a_n \neq 0$ and $b_m \neq 0$. Then
$$ T(r, R(f)) = dT(r, f) + S(r, f), $$
where $d = \max\{n, m\}$. 
3. Proof of the Theorem 1.2

Let \( F = \frac{L^m}{a} \) and \( G = \frac{(f^n)(a)}{a} \). Then \( F \) and \( G \) share \((1, l)\), except the zeros and poles of \( a(z) \). Let \( H \) be defined by (2.1)

Case 1. Let \( H \neq 0 \).

By our assumptions, \( H \) have poles only at zeros of \( F' \) and \( G' \) and poles of \( F \) and \( G \), and those 1-points of \( F \) and \( G \) whose multiplicities are distinct from the multiplicities of corresponding 1-points of \( G \) and \( F \) respectively. Thus, we deduce from (2.1) that

\[
N(r, H) \leq N\left(2, \frac{1}{H}\right) + N\left(1, \frac{1}{G}\right) + N\left(r, H\right)
+ N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + N_L\left(r, \frac{1}{F-1}\right)
+ N_L\left(r, \frac{1}{G-1}\right)
\]

(3.1)

here \( N_0\left(r, \frac{1}{F'}\right) \) is the counting function which only counts those points such that \( F' = 0 \) but \( F(F-1) \neq 0 \).

Because \( F \) and \( G \) share 1 IM, it is easy to see that

\[
N\left(r, \frac{1}{F-1}\right) = N_{11}\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + N_E^2\left(r, \frac{1}{G-1}\right)
\]

(3.2)

Using Lemma 2.2 and (3.1), (3.2) and (3.3) We get

\[
T(r, F) + T(r, G) \leq 3N\left(r, F\right) + N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right)
+ N_{11}\left(r, \frac{1}{F-1}\right) + 2N_E^2\left(r, \frac{1}{G-1}\right)
+ 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G)
\]

(3.4)

We discuss the following three sub cases.

Sub case 1.1. \( l \geq 2 \). Obviously.

\[
N_{11}\left(r, \frac{1}{F-1}\right) + 2N_E^2\left(r, \frac{1}{G-1}\right) + 3N_L\left(r, \frac{1}{F-1}\right) + 3N_L\left(r, \frac{1}{G-1}\right)
\]

\[
\leq N\left(r, \frac{1}{G-1}\right) + S(r, F)
\]

\[
\leq T(r, G) + S(r, F) + S(r, G)
\]

(3.5)
Combining (3.4) and (3.5), we get
\[ T(r, F) \leq 3N(r, F) + N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + S(r, F) \]  
(3.6)
that is
\[ T(r, f^m) \leq 3N(r, f^m) + N_2(r, \frac{1}{f^m}) + N_2(r, \frac{1}{(f^m)(k)}) + S(r, f) \]
By Lemma 2.1 for \( p = 2 \), we get
\[ mT(r, f) \leq (k + 5)\overline{N}(r, \frac{1}{f}) + (k + 4)\overline{N}(r, f) + S(r, f) \]
So
\[ (k + 4)\Theta(\infty, f) + (k + 5)\Theta(0, f) \leq 2k + 9 - m \]
which contradicts with (1.7).

**Sub case 2.** \( l = 1 \). It is easy to see that
\[ N_{11}(r, \frac{1}{F - 1}) + 2N^{[2]}_E(r, \frac{1}{G - 1}) + 2\overline{N}_L(r, \frac{1}{F - 1}) + 3\overline{N}_L(r, \frac{1}{G - 1}) \leq N(r, \frac{1}{G - 1}) + S(r, F) \]
\[ \leq T(r, G) + S(r, F) + S(r, G) \]
(3.7)
\[ \overline{N}_L(r, \frac{1}{F - 1}) \leq \frac{1}{2}N(r, \frac{E'}{F'}) \]
\[ \leq \frac{1}{2}N(r, \frac{F'}{F}) + S(r, F) \]
\[ \leq \frac{1}{2}[\overline{N}(r, \frac{1}{F}) + \overline{N}(r, F)] + S(r, F). \]
(3.8)
Combining (3.4) and (3.7) and (3.8), we get
\[ T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + \frac{7}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}(r, \frac{1}{F}) + S(r, F) \]
(3.9)
that is
\[ mT(r, f) \leq N_2(r, \frac{1}{f^m}) + N_2(r, \frac{1}{(f^m)(k)}) + \frac{7}{2}\overline{N}(r, f^m) + \frac{1}{2}\overline{N}(r, \frac{1}{f^m}) + S(r, f). \]
By Lemma 2.1 for \( p = 2 \), we get
\[ mT(r, f) \leq (k + \frac{9}{2})\overline{N}(r, f) + (k + \frac{11}{2})\overline{N}(r, \frac{1}{f}) + S(r, f) \]
So
\[ (k + \frac{9}{2})\Theta(\infty, f) + (k + \frac{11}{2})\Theta(0, f) \leq 2k + 10 - m \]
which contradicts with (1.8).

**Sub case 3.** \( l = 0 \). It is easy to see that
\[
\begin{align*}
N_{11}(r, \frac{1}{F - 1}) + 2N^{[2]}_E(r, \frac{1}{G - 1}) + \overline{N}_L(r, \frac{1}{F - 1}) + 2\overline{N}_L(r, \frac{1}{G - 1}) \leq N(r, \frac{1}{G - 1}) + S(r, F) \\
\leq T(r, G) + S(r, F) + S(r, F)
\end{align*}
\]
\[ \overline{N}_L(r, \frac{1}{F-1}) \leq N(r, \frac{1}{F-1}) - \overline{N}(r, \frac{1}{F-1}) \]
\[ \leq N(r, \frac{F'}{F}) \leq N(r, \frac{F'}{F}) + S(r, F) \]
\[ \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, F) + S(r, F). \] (3.11)

Similarly, we have
\[ \overline{N}_L(r, \frac{1}{G-1}) \leq \overline{N}(r, \frac{1}{G}) + \overline{N}(r, G) + S(r, F) \]
\[ \leq N_1(r, \frac{1}{G}) + \overline{N}(r, F) + S(r, G). \] (3.12)

Combining (3.4) and (3.10) – (3.12), we get
\[ T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 2\overline{N}(r, \frac{1}{F}) \]
\[ + 6\overline{N}(r, F) + N_1(r, \frac{1}{G}) + S(r, F) \] (3.13)

that is
\[ mT(r, f) \leq N_2(r, \frac{1}{f^m}) + N_2(r, \frac{1}{(f^n)(k)}) + 2\overline{N}(r, \frac{1}{f^m}) \]
\[ + 6\overline{N}(r, \frac{1}{f^m}) + N_1(r, \frac{1}{(f^n)(k)}) + S(r, f). \]

By Lemma 2.1 for \( p = 2 \) and for \( p = 1 \) respectively, we get
\[ mT(r, f) \leq (2k + 8)\overline{N}(r, \frac{1}{F}) + (2k + 7)\overline{N}(r, f). \]

So
\[ (2k + 7)\Theta(∞, f) + (2k + 8)\Theta(0, f) \leq 4k + 15 - m \]
which contradicts with (1.9).

**Case 2.** Let \( H \equiv 0 \).

on integration we get from (2.1)
\[ \frac{1}{F-1} = \frac{C}{G-1} + D, \] (3.14)

where \( C, D \) are constants and \( C \neq 0 \). we will prove that \( D = 0 \).

**Sub case 2.1.** Suppose \( D \neq 0 \). If \( z_0 \) be a pole of \( f \) with multiplicity \( p \) such that \( a(z_0) \neq 0, ∞ \), then it is a pole of \( G \) with multiplicity \( np + k \) respectively. This contradicts (3.14). It follows that \( N(r, f) = S(r, f) \) and hence \( \Theta(∞, f) = 1 \). Also it is clear that \( \overline{N}(r, f) = \overline{N}(r, G) = S(r, f) \). From (1.7)-(1.9) we know respectively
\[ (k + 5)\Theta(0, f) > k + 5 - m \] (3.15)
\[ (k + \frac{11}{2})\Theta(0, f) > k + \frac{11}{2} - m \] (3.16)

and
\[ (2k + 8)\Theta(0, f) > 2k + 8 - m \] (3.17)

Since \( D \neq 0 \), from (3.14) we get
\[ \overline{N} \left( r, \frac{1}{F - (1 + \frac{1}{F})} \right) = \overline{N}(r, G) = S(r, f) \]
Suppose $D \neq -1$.

Using the second fundamental theorem for $F$ we get

$$T(r, F) \leq \mathcal{N}(r, F) + \mathcal{N}(r, \frac{1}{F}) + \mathcal{N}\left(r, \frac{1}{F - (1 + \frac{1}{D})}\right)$$

$$\leq \mathcal{N}(r, \frac{1}{F}) + S(r, f)$$

i.e.,

$$mT(r, F) \leq \mathcal{N}(r, \frac{1}{F}) + S(r, f)$$

$$\leq mT(r, f) + S(r, f).$$

So, we have $mT(r, f) = \mathcal{N}(r, \frac{1}{f})$ and so $\Theta(0, f) = 1 - m$. Which contradicts (3.15) – (3.17).

If $D = -1$, then

$$\frac{F}{F - 1} = C \frac{1}{G - 1}$$

and from which we know $\mathcal{N}(r, \frac{1}{f}) = \mathcal{N}(r, G) = S(r, f)$ and hence, $\mathcal{N}(r, \frac{1}{f}) = S(r, f)$.

If $C \neq -1$,

we know from (3.18) that

$$\mathcal{N}\left(r, \frac{1}{G - (1 + C)}\right) = \mathcal{N}(r, F) = S(r, f).$$

So from Lemma 2.1 and the Second fundamental theorem we get

$$T(r, (f^n)^{(k)}) \leq \mathcal{N}(r, G) + \mathcal{N}(r, \frac{1}{G}) + \mathcal{N}\left(r, \frac{1}{G - (1 + C)}\right) + S(r, f)$$

$$\leq \mathcal{N}\left(r, \frac{1}{(f^n)^{(k)}}\right) + S(r, f)$$

$$mT(r, f) \leq (k + 1)\mathcal{N}(r, \frac{1}{f}) + k\mathcal{N}(r, f) + S(r, f),$$

which is absurd. So $C = -1$ and we get from (3.18) that $FG \equiv 1$, which implies

$$\left[\frac{(f^n)^{(k)}}{f^n}\right] = \frac{n^2}{f^{n+m}}.$$ 

In view of the first fundamental theorem, we get from above

$$(n + m)T(r, f) \leq k[\mathcal{N}(r, f) + \mathcal{N}(r, \frac{1}{f})] + S(r, f) = S(r, f),$$

which is impossible.

**Sub case 2.2.** $D = 0$ and so from (3.14) we get

$$G - 1 \equiv C(F - 1).$$

If $C \neq 1$, then

$$G \equiv C(F - 1 + \frac{1}{C})$$

and $\mathcal{N}(r, \frac{1}{G}) = \mathcal{N}\left(r, \frac{1}{F - (1 + \frac{1}{C})}\right)$. 

By the second fundamental theorem and Lemma 2.1 for $p = 1$ and Lemma 2.3 we have

$$mT(r, f) + S(r, f) = T(r, F)$$

$$\leq N(r, F) + N(r, f) + \left(r, \frac{1}{F - (1 - \frac{1}{e})}\right) + S(r, G)$$

$$\leq N(r, f^m) + N(r, f) + N\left(r, \frac{1}{(f^m)k}\right) + S(r, f)$$

$$\leq N(r, f) + N\left(r, \frac{1}{f}\right) + (k + 1)N\left(r, \frac{1}{f}\right) + kN(r, f) + S(r, f)$$

$$\leq (k + 2)N\left(r, \frac{1}{f}\right) + (k + 1)N(r, f) + S(r, f).$$

Hence

$$(k + 1)\Theta(\infty, f) + (k + 2)\Theta(0, f) \leq 2k + 3 - m.$$ 

So, it follows that

$$(k + 4)\Theta(\infty, f) + (k + 5)\Theta(0, f) \leq 3\Theta(\infty, f) + (k + 1)\Theta(\infty, f)$$

$$+ (k + 3)\Theta(0, f) + 2\Theta(0, f)$$

$$\leq 2k + 9 - m$$

$$(k + \frac{9}{2})\Theta(\infty, f) + (k + \frac{11}{2})\Theta(0, f) \leq 2k + 10 - m,$$

and

$$(2k + 7)\Theta(\infty, f) + (2k + 8)\Theta(0, f) \leq 4k + 15 - m.$$ 

This contradicts (1.7) - (1.9). Hence $C = 1$ and so $F \equiv G$, that is $f^m \equiv (f^n)^{(k)}$. This completes the proof of the theorem.

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References


HARINA P. WAGHAMORE  
Department of Mathematics, Central College Campus, Bangalore University, Bangalore-560 001, INDIA  
*E-mail address*: harinapw@gmail.com

RAJESHWARI S.  
Department of Mathematics, Central College Campus, Bangalore University, Bangalore-560 001, INDIA  
*E-mail address*: rajeshwaripreetham@gmail.com