Real hypersurface of a complex space form

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Abstract: The purpose of the present paper is to give characterization of real hypersurface of a complex space form. We find conditions for these hypersurfaces to be $\phi$-symmetric and to have $\eta$-parallel curvature tensor. Further we prove totally $\eta$-umbilical real hypersurfaces of complex space forms have $\xi$-parallel Ricci tensor and $\xi$-parallel structure Jacobi operator.

Keywords: Complex space form, real hypersurface, $\eta$-parallel, $\eta$ umbilical, $\xi$-parallel, structure Jacobi operator.

1 Introduction

A complex $n(\geq 2)$-dimensional Kählerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. The induced almost contact metric structure of a real hypersurface $M$ of $M_n(c)$ is denoted by $(\phi, \xi, \eta, g)$.

In [1] and [2] Berndt has called real hypersurface in $M_n(c)$ with the principal vector $\xi$ as Hopf real hypersurface. It can be easily seen that there does not exist any real hypersurface in $M_n(c)$, $c \neq 0$, which is locally symmetric that is $VR = 0$. This motivates the introduction of the notion of $\eta$-parallel curvature tensor (Lee J.G et al [9]). The notion of $\eta$-parallel curvature tensor is defined by $g((\nabla_X R)(Y,Z)U,V) = 0$ for any $X,Y,Z,U$ and $V$ in a distribution orthogonal to $\xi$. This notion is weaker to the notion of $\eta$-parallel second fundamental tensor $g((\nabla_X A)Y,Z) = 0$ for any $X,Y$ and $Z$.

2 preliminaries

Let $M_n(c)$ denote the complex space form of complex dimension $n$ with constant holomorphic sectional curvature $4c$. Let $M$ be a real $(2n - 1)$-dimensional hypersurface immersed in $M_n(c)$ with parallel almost complex structure $J$ and $N$ be unit normal vector field on $M$. For any vector field $X$ tangent to $M$, we define $\phi$, $\eta$ and $\xi$ by

$$JX = \phi(X)N, JN = -\xi,$$ (1)

where $\phi X$ is the tangential part of $JX$, $\phi$ is a tensor field of a type $(1,1)$, $\eta$ is a 1-form, and $\xi$ is the unit vector field on $M$. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \phi \xi = 0,$$
$$\eta(\xi) = 1, g(X,\xi) = \eta(X), \eta(\phi X) = 0,$$
$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$g(\phi X, Y) = -g(X,\phi Y), g(\phi X, X) = 0,$$
Thus \((\phi, \xi, \eta, g)\) is an almost contact metric structure on \(M\). We denote by \(\tilde{\nabla}\) the operator of covariant differentiation in \(M_n(c)\) and by \(\nabla\) the one in \(M\) determined by the induced metric. Then the Gauss and Weingarten formulae are given respectively by

\[
\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \\
\tilde{\nabla}_X N = -AX,
\]

for any vector fields \(X\) and \(Y\) tangent to \(M\). We call \(A\) the shape operator of \(M\). For the almost contact metric structure on \(M\), we have

\[
\nabla_X \xi = \phi AX,
\]

\[
(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.
\]

We denote by \(R\) the Riemannian curvature tensor field of \(M\). Then the equation of Gauss is given by

\[
R(X, Y)Z = \frac{c}{4}[g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY
\]

and the equation of Codazzi is

\[
(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi].
\]

If the shape operator \(A\) of \(M\) is of the form \(AX = aX + b\eta(X)\xi\) for some functions \(a\) and \(b\), then \(M\) is said to be totally \(\eta\)-umbilical. We put \(\alpha = g(A\xi, \xi)\). If \(\xi\) is a principal vector everywhere (i.e. \(A\xi = \alpha\xi\)), we say that \(M\) is a Hopf hypersurface and \(M\) is called a hypersurface with recurrent shape operator if there exists a 1-form \(\alpha\) such that \(A\) of \(M\) satisfies \((\nabla_X A)Z = \alpha(X)AZ\).

3 Real hypersurfaces of complex space forms with recurrent second fundamental form

Let \(M\) be a real hypersurface in a complex space form \(M_n(c)\), \(c \neq 0\). Taking covariant derivative of \((7)\) with respect to \(X\), we get

\[
(\nabla_X R)(Y, Z)U = \frac{c}{4}[\eta(Z)g(AX, U)\phi Y - \eta(U)g(AX, Z)\phi Y + \eta(Y)g(\phi Z, U)AX \\
- g(OX, Y)g(\phi Z, U)\xi - \eta(Y)g(AX, U)\phi Z + \eta(U)g(AX, Y)\phi Z - \eta(Z)g(\phi Y, U)AX \\
+ g(OX, Z)g(\phi Y, U)\xi - 2\eta(Y)g(AX, Z)\phi U + 2\eta(Z)g(AX, Y)\phi U - 2\eta(U)g(\phi Y, Z)AX \\
+ g(OX, U)g(\phi Y, Z)\xi] + g((\nabla_X A)Z, U)AY + (\nabla_X A)Y g(AX, U) - g((\nabla_X A)Y, U)AZ \\
- g((\nabla_X A)Z, U)AZ.
\]
Contracting the above equation with respect to $W$, we get
\begin{align}
g((\nabla_X R)(Y, Z)U, W) &= \frac{c}{4} [\eta(Z)g(AX, U)g(\phi Y, W) - \eta(U)g(AX, Z)g(\phi Y, W) \\
&+ \eta(Y)g(AX, W)g(\phi Z, U) - \eta(W)g(AX, Y)g(\phi Z, U) - \eta(Y)g(AX, U)g(\phi Z, W) \\
&+ \eta(U)g(AX, Y)g(\phi Z, W) - \eta(Z)g(AX, W)g(\phi Y, U) + \eta(W)g(AX, Z)g(\phi Y, U) \\
&- 2\eta(Y)g(AX, Z)g(\phi U, W) + 2\eta(Z)g(AX, Y)g(\phi U, W) - 2\eta(U)g(AX, W)g(\phi Y, Z) \\
&+ 2\eta(W)g(AX, U)g(\phi Y, Z)] + g((\nabla_X A)Z, U)g(AY, W) + g((\nabla_X A)Y, W)g(AZ, U) \\
&- g((\nabla_X A)Y, U)g(AZ, W) - g((\nabla_X A)Z, U)g(AZ, W).
\end{align}

If the shape operator $A$ is recurrent, then we have
\begin{align}
g((\nabla_X R)(Y, Z)U, W) &= \frac{c}{4} [\eta(Z)g(AX, U)g(\phi Y, W) - \eta(U)g(AX, Z)g(\phi Y, W) \\
&+ \eta(Y)g(AX, W)g(\phi Z, U) - \eta(W)g(AX, Y)g(\phi Z, U) - \eta(Y)g(AX, U)g(\phi Z, W) \\
&+ \eta(U)g(AX, Y)g(\phi Z, W) - \eta(Z)g(AX, W)g(\phi Y, U) + \eta(W)g(AX, Z)g(\phi Y, U) \\
&- 2\eta(Y)g(AX, Z)g(\phi U, W) + 2\eta(Z)g(AX, Y)g(\phi U, W) - 2\eta(U)g(AX, W)g(\phi Y, Z) \\
&+ 2\eta(W)g(AX, U)g(\phi Y, Z)] + 2\alpha(X)g(AZ, U)g(AY, W) - g(AY, U)g(AZ, W).
\end{align}

For all $X, Y, Z, U, W$ orthogonal to $\xi$, the above equation reduces to
\begin{align}
g((\nabla_X R)(Y, Z)U, W) = 2\alpha(X)[g(AZ, U)g(AY, W) - g(AY, U)g(AZ, W)].
\end{align}

Thus we have

**Proposition 1.** A real hypersurface of a complex space form with recurrent second fundamental form has $\eta$-parallel curvature tensor if and only if
\[ [g(AZ, U)g(AY, W) - g(AY, U)g(AZ, W)] = 0 \text{ holds.} \]

Applying $\phi^2$ to (1), we get
\begin{align}
\phi^2((\nabla_X R)(Y, Z)U) &= \frac{c}{4} [-\eta(Z)g(AX, U)\phi Y + \eta(U)g(AX, Z)\phi Y - \eta(Y)g(\phi Z, U)AX \\
&+ \eta(Y)\eta(AX)g(\phi Z, U)\xi + \eta(Y)g(AX, U)\phi Z - \eta(U)g(AX, Y)\phi Z + \eta(Z)g(\phi Y, U)AX \\
&- \eta(Z)\eta(AX)g(\phi Y, U)\xi + 2\eta(Y)g(AX, Z)\phi U - 2\eta(U)g(AX, Y)\phi U + 2\eta(U)g(\phi Y, Z)AX \\
&- 2\eta(U)\eta(AX)g(\phi Y, Z)\xi - g((\nabla_X A)Z, U)AY + \eta(AY)g((\nabla_X A)Z, U)\xi - g(AZ, U)((\nabla_X A)Y) \\
&+ \eta((\nabla_X A)Y)g(AZ, U)\xi + g((\nabla_X A)Y, U)AZ - \eta(AZ)g((\nabla_X A)Y, U)\xi + g(AY, U)((\nabla_X A)Z) \\
&- \eta((\nabla_X A)Z)g(AY, U)\xi].
\end{align}

If the shape operator $A$ is recurrent, then (10) takes the form
\begin{align}
\phi^2((\nabla_X R)(Y, Z)U) &= \frac{c}{4} [-\eta(Z)g(AX, U)\phi Y + \eta(U)g(AX, Z)\phi Y - \eta(Y)g(\phi Z, U)AX \\
&+ \eta(Y)\eta(AX)g(\phi Z, U)\xi + \eta(Y)g(AX, U)\phi Z - \eta(U)g(AX, Y)\phi Z + \eta(Z)g(\phi Y, U)AX \\
&- \eta(Z)\eta(AX)g(\phi Y, U)\xi + 2\eta(Y)g(AX, Z)\phi U - 2\eta(U)g(AX, Y)\phi U + 2\eta(U)g(\phi Y, Z)AX \\
&- 2\eta(U)\eta(AX)g(\phi Y, Z)\xi - g((\nabla_X A)Z, U)AY + \eta(AY)g((\nabla_X A)Z, U)\xi - g(AZ, U)((\nabla_X A)Y) \\
&+ \eta((\nabla_X A)Y)g(AZ, U)\xi + g((\nabla_X A)Y, U)AZ - \eta(AZ)g((\nabla_X A)Y, U)\xi + g(AY, U)((\nabla_X A)Z) \\
&- \eta((\nabla_X A)Z)g(AY, U)\xi - 2\alpha(X)[g(AZ, U)AZ - g(AZ, U)AY].
\end{align}
For all $X,Y,Z,U,W$ orthogonal to $\xi$, the above equation reduces to

$$\phi^2((\nabla X)R)(Y,Z)U = 2\alpha(X)[g(AY,U)AZ - g(AZ,U)AY].$$  \hspace{1cm} (12)

Thus we have

**Proposition 2.** A real hypersurface of a complex space form with recurrent second fundamental form is $\phi$ -symmetric if and only if $[g(AZ,U)g(AY,W) - g(AY,U)g(AZ,W)] = 0$ holds.

The condition $\phi$ -symmetry on $R$ is weaker to $\eta$ -parallel condition on $R$. But these two conditions are equivalent as stated in the following. Combining proposition 1 and proposition 2, we can state the following:

**Theorem 1.** In a real hypersurface $M$ of a complex space form $M_n(c)$ with recurrent second fundamental form $A$ satisfying the condition $[g(AZ,U)g(AY,W) - g(AY,U)g(AZ,W)] = 0$, the following are equivalent:

(i) $M$ has $\eta$ -parallel curvature tensor.

(ii) $M$ is $\phi$ -symmetric.

## 4 Totally $\eta$-umbilical real hypersurfaces of complex space forms

Let $M$ be a totally $\eta$-umbilical hypersurface of the complex space form $M_n(c)$. Then using (10), we get

$$g((\nabla X)R)(Y,Z)U,W) = \frac{c}{4}[\eta(Z)g(AX,U)g(\phi Y,W) - \eta(U)g(AX,Z)g(\phi Y,W)$$
$$+ \eta(Y)g(AX,W)g(\phi Z,U) - \eta(W)g(AX,Y)g(\phi Z,U)$$
$$- \eta(Y)g(AX,U)g(\phi Z,W) + \eta(U)g(AX,Y)g(\phi Z,W)$$
$$- \eta(Z)g(AX,W)g(\phi Y,U) + \eta(W)g(AX,Z)g(\phi Y,U)$$
$$- 2\eta(Y)g(AX,Z)g(\phi U,W) + 2\eta(AX)g(AY,X)g(\phi U,W)$$
$$- 2\eta(U)g(AX,W)g(\phi Y,Z) + 2\eta(W)g(AU,X)g(\phi Y,Z)$$
$$+ (Xa)g(AY,Z)g(aY,W) + (Xa)\eta(Y)g(AY,Z)g(aY,W)$$
$$+ (Xa)\eta(Z)g(aY,W) + b\eta(U)g(\phi AX,Z)g(aY,W)$$
$$+ b\eta(Z)g(\phi AX,U)g(aY,W) + (Xa)g(aZ,U)g(\phi Y,W)$$
$$+ (Xa)\eta(Y)g(aZ,U)g(\phi Y,W) + b\eta(Y)g(\phi AX,W)g(aZ,U) + b\eta(Z)g(\phi AX,U)g(aZ,U)$$
$$- (Xa)g(Y,U)g(aZ,W) - (Xb)\eta(Y)g(\phi Y,U)g(aZ,W)$$
$$- b\eta(U)g(\phi AX,Y)g(aZ,W) - b\eta(Y)g(\phi AX,U)g(aZ,W)$$
$$- b\eta(Y)g(\phi AX,U)g(aZ,W) - b\eta(U)g(\phi AX,Y)g(aZ,W)$$
$$- (Xa)g(Z,W)g(aY,U) - (Xb)\eta(Z)g(W)g(aY,U)$$
$$- b\eta(Z)g(\phi AX,U)g(aY,U) - b\eta(Z)g(\phi AX,W)g(aY,U)$$
$$- b(Xa)\eta(Y)g(\phi Y,W).$$  \hspace{1cm} (13)

Then for all $X,Y,Z,U,W$ orthogonal to $\xi$, we have

$$g((\nabla X)R)(Y,Z)U,W) = 0.$$  \hspace{1cm} i.e. $M$ has $\eta$ -parallel curvature tensor.

Thus we have

**Theorem 2.** A totally $\eta$-umbilical real hypersurface of a complex space form has $\eta$ -parallel curvature tensor.
Now from (1), for a totally $\eta$-umbilical real hypersurface, we have
\[
\phi^2(\nabla_X R)(Y, Z)U = \frac{c}{4} \{- \eta(Z)g(AX, U)\phi Y + \eta(U)g(AX, Z)\phi Y - \eta(Y)g(\phi Z, U)AX \\
+ \eta(Y)g(\phi Z, U)\xi + \eta(Y)g(AX, U)\phi Z - \eta(U)g(AX, Y)\phi Z + \eta(Z)g(\phi Y, U)AX \\
- \eta(Z)g(\phi Y, U)\xi + 2\eta(Y)g(AX, Z)\phi U - 2\eta(Z)g(AX, Y)\phi U + 2\eta(U)g(\phi Y, Z)AX \\
- 2\eta(U)g(AX, g(\phi Y, Z)\xi) - (Xa)g(Z, U)aY - (Xb)\eta(Z)\eta(U)aY - b\eta(U)g(\phi AX, Z)aY \\
- b(Xa)\eta(Y)g(Z, U)\xi + \eta(\eta(Y)(Xa))g(Z, U)\xi + (Xb)\eta(Z)\eta(U)\eta(\eta(Y)\xi) - g(aZ, U)(Xa)Y \\
- b\eta(Y)g(aZ, U)aAX - b\eta(Z)\eta(U)(Xa)Y + \eta(U)g(aZ, U)\xi + (Xa)g(Y, U)aZ \\
+ (Xb)\eta(U)\eta(Y)aZ + b\eta(U)g(\phi AX, Y)aZ + \eta(Y)g(\phi AX, U)aZ - b\eta(U)\eta(\eta(Z)g(\phi AX, U)\xi \\
- (Xb)\eta(aZ)\eta(U)\eta(\eta(Z)g(\phi AX, Y)\xi - b\eta(Y)\eta(aZ)g(\phi AX, U)\xi \\
+ g(aY, U)(Xa)Z + \eta(\eta(Z)g(aY, U)aAX + b\eta(Y)\eta(U)(Xa)Z - \eta(Z)g(aY, U)(Xa)\xi.
\]

For all $X, Y, Z, U, W$ orthogonal to $\xi$, the above equation reduces to
\[
\phi^2(\nabla_X R)(Y, Z)U = -(Xa)g(Z, U)aY - g(AZ, U)(Xa)Y + (Xa)g(Y, U)aZ + g(AY, U)(Xa)Z.
\]

It is well known that:

**Remark 1.** Any totally $\eta$-umbilical real hypersurface is a Hopf hypersurface.

**Remark 2.** In a Hopf hypersurface with $\xi$ as a principal vector, principal curvature corresponding to $\xi$ is a constant.

From the above remarks we have for constant $a$,
\[
\phi^2(\nabla_X R)(Y, Z)U = 0.
\]

Thus we can state that

**Theorem 3.** A totally $\eta$-umbilical real hypersurface of a complex space form is $\phi$ -symmetric.

Now let us denote by $Q$ the Ricci tensor of $M$ in $M_\eta(c)$. Then from (7) which together with (5), we obtain
\[
QX = \frac{c}{4} \{(2n + 1)X - 3\eta(x)\xi\} + hAX - A^2X,
\]
\[
(\nabla_X Q)Y = -\frac{3c}{4} [g(\phi AX, Y)\xi + \eta(Y)\phi AX] + (Xh)AY + (hI - A)(\nabla_X Y) - (\nabla_X A)AY, \tag{14}
\]
where $I$ denote the identity map on the tangent space $T_pM$, $p \in M$.

If $M$ is totally $\eta$-umbilical, then from (4) we obtain
\[
(\nabla_X Q)Y = (\xi h)c\xi + (hI - A)((\xi a)Y + (\xi b)\eta(Y)\xi) - ((\xi a)AY + (\xi b)\eta(AY)\xi),
\]
By remark 1 and remark 2, we have $(\nabla_X Q)Y = 0$, provided $a = -b$.

**Theorem 4.** A totally $\eta$-umbilical real hypersurface of a complex space form has $\xi$ -parallel Ricci tensor provided $a = -b$. 
From (5), we obtain
\[ R\xi Y = R(Y, \xi)\xi = \frac{c}{4} \{ Y - \eta(Y)\xi \} + \alpha AY - \eta(AY)A\xi \] (15)
for any vector field \( Y \) on \( M \). Taking covariant derivative with respect to \( X \) of (15), we get
\[
\langle \nabla_X R\xi Y \rangle = -\frac{c}{4} \{ \langle \nabla_X \eta \rangle Y \} - \frac{c}{4} \{ \eta(Y) \langle \nabla_X \xi \rangle + \alpha AY \}
\]
\[
+\eta(\langle \nabla_X A \rangle Y) - \langle \langle \nabla_X \eta \rangle AY \rangle A\xi - \eta(\langle \nabla_X A \rangle Y)A\xi
\]
\[-\eta(AY)\langle \nabla_X \xi \rangle - \eta(AY)A\langle \nabla_X \xi \rangle \]
Contracting the above with respect to \( Z \) and using (4) and (5), we get
\[
\begin{align*}
g(\langle \nabla_X R\xi \rangle Y, Z) &= -\frac{c}{4} [g(\langle \nabla_X \xi \rangle Y, \eta(Z)) + \eta(Y)g(\langle \nabla_X \xi \rangle, Z)] \\
&\quad + X\alpha g(AY, Z) + \alpha g(\langle \nabla_X A \rangle Y, Z) + g(X, A\phi AY)g(A\xi, Z) \\
&\quad - g(\langle \nabla_X A \rangle Y, \xi)g(AX, Z) - g(AY, \xi)g(\langle \nabla_X A \rangle \xi, Z) + g(AY, \xi)g(X, A\phi AY).
\end{align*}
\]
Since \( M \) is totally \( \eta \)-umbilical, the above equation reduces to
\[
\begin{align*}
g(\langle \nabla_X R\xi \rangle Y, Z) &= -\frac{c}{4} [g(\phi AX, Y)\eta(Z) + \eta(Y)g(\phi AX, Z)] + (X\alpha) [g(AY, Z) \\
&\quad + b(\eta(Y)\eta(Z) + \alpha (Xa)g(AY, Z) + (Xb)\eta(Y)\eta(Z) + b(g(\phi AX, Y)\eta(Z) \\
&\quad + \eta(Y)g(\phi AX, Z)] + \eta(AX)g(X, A\phi AY) - (Xa)\eta(Y) \\
&\quad + (Xb)\eta(Y) + bg(\phi AX, Y))\eta(AX) - (Xa)\eta(Z) \\
&\quad + (Xb)\eta(Z) + bg(\phi AX, Z))\eta(AY) + \eta(AY)g(X, A\phi AY).
\end{align*}
\]
For all \( X, Y, Z \) orthogonal to \( \xi \), we obtain
\[
g(\langle \nabla_X R\xi \rangle Y, Z) = (X\alpha)g(AY, Z) + \alpha (Xa)g(Y, Z).
\]
If \( a \) and \( b \) are constants, then we have
\[
g(\langle \nabla_X R\xi \rangle Y, Z) = 0.
\]

**Theorem 5.** A totally \( \eta \)-umbilical real hypersurface of a complex space form has \( \xi \) -parallel structure Jacobi operator provided the associated scalars \( a \) and \( b \) are constants.

**References**


[7] Byung Hak Kim and Sadahoro Maeda, Totally $\eta$-umbilic hypersurfaces in a nonflat complex space form and their almost contact metric structures, Scientiae Mathematicae japonicae online, e-2010, 483-490.


[11] Yumetaro Mashiko, Satoshi Kurosu and Yoshio Matsuyama, On A Kaehler hypersurface of a complex space form with the recurrent second fundamental form,