Bénard–Marangoni ferroconvection with magnetic field dependent viscosity

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Abstract

The effect of magnetic field dependent viscosity on the onset of Bénard–Marangoni ferroconvection in a horizontal layer of ferrofluid is investigated theoretically. The lower boundary is taken to be rigid with fixed temperature, while the upper free boundary at which temperature-dependent surface tension effect is considered is non-deformable and subject to a general thermal condition. The Rayleigh–Ritz method with Chebyshev polynomials of the second kind as trial functions is employed to extract the critical stability parameters numerically. The results show that the onset of ferroconvection is delayed with an increase in the magnetic field dependent viscosity parameter (λ) and Biot number (Bi) but opposite is the case with an increase in the value of magnetic Rayleigh number (Rm) and nonlinearity of magnetization (M_s). Further, increase in Rm, M_s, and decrease in λ and Bi is to decrease the size of the convection cells.

1. Introduction

A typical ferromagnetic fluid contains single domain nanoparticles of magnetic material (iron, cobalt or magnetite) stably suspended in a liquid carrier with low electrical conductivity. Each particle is encapsulated by a monolayer of surfactant in order to prevent particle coalescence due to magnetic attraction. The average size of magnetic nanoparticles is about 10 nm. Magnetic colloids have magnetic susceptibility which is thousands times larger than that of natural materials. Such fluids became the subject of a special branch of magnetohydrodynamics termed as ferrohydrodynamics (Rosensweig [11]) and found applications in various areas of science, technology and nanotechnology (Bashtovoy et al. [2, 3]).

The magnetization of ferromagnetic fluids depends on the magnetic field, the temperature and the density of the fluid. Any variation of these quantities can induce a change in body force distribution in the fluid. This leads to convection in ferrofluids in the presence of magnetic field gradient, known as ferroconvection, which is similar to buoyancy driven convection. Buoyancy driven convection in a layer of ferrofluid heated uniformly from below in the presence of a uniform magnetic field has been studied extensively over the years. In the first theoretical study (Finlayson [4]), which dealt with convection in a horizontal layer of magnetic fluid subject to a vertical temperature gradient and placed in a transverse uniform magnetic field, the concentration of magnetic particles was assumed to be constant. Therefore only thermo-gravitational and thermomagnetic mechanisms of convection were considered. The discussed theory predicted a destabilizing influence of the magnetic field and extended continues over the years (Lalas and Carmi [5]; Shliomis [6]; Gotoh and Yamada [7]; Stiles and Kagan [8]; Kaloni and Lou [9]). The non-linear stability analysis for a magnetized ferrofluid layer heated from below for stress-free boundaries has been performed by Sunil and Mahjan [10]. A variety of velocity and temperature boundary conditions on the onset of ferroconvection in an initially quiescent ferrofluid layer has been considered by Nanjundappa and Shivakumara [11]. Recently, thermal convection of ferrofluids in the presence of a uniform vertical magnetic field with the boundary temperatures modulated sinusoidally about some reference values has been discussed by Singh and Bajaj [12], while Belyaev and Smorodin [13] have studied the effect of an alternating uniform magnetic field on the onset of convection in a horizontal layer of a ferrofluid within the framework of a quasi-stationary approach.

It is a well established fact that convection can also be induced by surface-tension forces provided it is a function of temperature. In view of the fact that heat transfer is greatly enhanced due to convection, the magnetic convection problems offer new possibilities for new applications in cooling with motors, loud speakers, transmission lines, and other equipment where magnetic field is already present. If the ferrofluid layer has an upper surface open to atmosphere then the instability is due to
the combined effects of the buoyancy as well as temperature-dependent surface tension forces, known as Bénard–Marangoni ferroconvection. A limited number of studies have addressed the effect of surface tension forces on ferroconvection in a horizontal ferrofluid layer. Linear and non-linear stability of combined buoyancy-surface tension effects in a ferrofluid layer heated from a ferrofluid layer. Linear and non-linear stability of combined effect of surface tension forces on ferroconvection in a horizontal ferrofluid layer in the presence of weak vertical magnetic field normal to the boundaries has been discussed by Hennenberg et al. [17]. The onset of Marangoni ferroconvection with different initial temperature conditions.

Thermal convection in ferromagnetic fluids is gaining much importance due to its astounding physical properties. One such property is viscosity of the ferromagnetic fluid. The viscosity of the ferrofluid is predicted by dimensional analysis to be a function of the ratio of hydrodynamic stress to magnetic stress (Rosenswieg et al. [19]). The effect of a homogeneous magnetic field on the viscosity of a fluid with solid particles possessing intrinsic magnetic moments has been investigated by Shiomi et al. [20].

The effect of magnetic field dependent (MFD) viscosity on the onset of ferroconvection in a rotating ferrofluid layer is discussed by Vaidyanathan et al. [21], with or without dust particles by Sunil et al. [22] and the non-linear stability analysis has also been performed by Sunil et al. [23]. Recently, Nanjundappa et al. [24] have investigated the effect of MFD viscosity on the onset of convection in a ferromagnetic fluid layer in the presence of a vertical magnetic field by considering the bounding surfaces are either rigid-ferromagnetic or stress-free with constant heat flux conditions.

The intent of the present paper is to study coupled Bénard–Marangoni ferroconvection in a ferrofluid layer in the presence of a uniform vertical magnetic field with magnetic field dependent viscosity. The lower boundary is rigid with fixed temperature, while the upper non-deformable free boundary is subjected to temperature dependent surface tension forces and a general thermal boundary condition on the perturbation temperature is imposed. The study helps in understanding control of ferroconvection by magnetic field dependent viscosity, which is useful in many heat transfer related problems particularly in materials science processing. The resulting eigenvalue problem is solved numerically by employing the Rayleigh–Ritz method with Chebyshev polynomials of the second kind as trial functions.

The paper is organized as under. Section 2 is devoted to the formulation of the problem. The method of solution is discussed in Section 3. In Section 4, the numerical results are discussed and some important conclusions follow in Section 5.

2. Mathematical formulation

We consider a Boussinesq ferrofluid layer of thickness $d$ with no lateral boundaries and a uniform magnetic field $H_0$ acting normal to the boundaries. The lower and the upper boundaries are maintained at constant but different temperatures $T_0$ and $T_1( < T_0)$, respectively. A Cartesian co-ordinate system $(x, y, z)$ is used with the origin at the lower boundary and the $z$-axis vertically upward. Gravity acts in the negative $z$-direction, $\vec{g} = -g \hat{k}$, where $\hat{k}$ is the unit vector in the $z$-direction. The layer is bounded below by a rigid surface while the free surface which is subjected to temperature dependent surface tension forces is assumed to be flat and non-deformable. The surface tension $\sigma$ is assumed to vary linearly with temperature in the form

$$\sigma = \sigma_0 - \sigma_1(T - T_0)$$

where $\sigma_0$ is the unperturbed value and $-\sigma_1$ is the rate of change of surface tension with temperature. The fluid density $\rho$ is assumed to vary linearly with temperature in the form

$$\rho = \rho_0 [1 - \alpha_0 (T - T_0)]$$

where $\alpha_0$ is the thermal expansion coefficient and $\rho_0$ is the density at $T = T_0$.

In the study of ferroconvection, we have to solve the Maxwell equations simultaneously with the balance of mass, linear momentum and energy. Since the fluid is assumed to be electrically not conducting, the Maxwell equations reduce to

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = 0$$

where $\vec{B}$ is the magnetic induction and $\vec{H}$ the intensity of magnetic field. In view of Eq. (4), we can express the magnetic field by a scalar potential

$$\vec{H} = \nabla \phi$$

Further $\vec{B}$, $\vec{M}$ and $\vec{H}$ are related by

$$\vec{B} = \mu_0 (\vec{M} + \vec{H})$$

where $\vec{M}$ is the magnetization and $\mu_0$ the magnetic permeability of vacuum.

Following Finlayson [4], we assume that the magnetization is aligned with the magnetic field, but allow dependence on the magnitude of magnetic field as well as on the temperature in the form

$$\vec{M} = [M_0 + \chi (H - H_0) - K (T - T_0)] \frac{\vec{H}}{H}$$

where $M_0 = M(H_0, T_0)$, $H = |\vec{H}|$, $M = |\vec{M}|$, $\chi = (\partial M/\partial H)_{H_0, T_0}$ is the magnetic susceptibility and $K = -(\partial M/\partial T)_{H_0, T_0}$ is the pyromagnetic coefficient.

The momentum equation is

$$\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla \cdot (\vec{q} \cdot \vec{v} \vec{q}) = -\nabla p + \rho \vec{g} + \mu_0 (\vec{M} \cdot \vec{v}) \vec{H} + 2 \nabla \cdot \left[ \frac{\eta \vec{D}}{2} \right]$$

where $\vec{q} = (u, v, w)$ is the velocity, $p$ the pressure, $t$ the time and $\vec{D} = (\nabla \vec{q} + (\nabla \vec{q})^T)/2$ the rate of strain tensor. The fluid is assumed to be incompressible having variable viscosity. Experimentally, it has been demonstrated that the magnetic viscosity has got exponential variation, with respect to magnetic field (Rosenswieg et al. [19]). As a first approximation, for small field variation, linear variation of magnetic viscosity has been used in the form

$$\eta = \eta_0 (1 + \beta \cdot \vec{B})$$

where $\beta$ is the variation coefficient of magnetic field dependent viscosity and is considered to be isotropic (Vaidyanathan et al. [21]), \( \eta_0 \) is taken as viscosity of the fluid when the applied magnetic field is absent.

Neglecting viscous dissipation, the energy equation is [4]

$$[\rho_0 C_{V,H} - \mu_0 \vec{H} \cdot \left( \frac{\partial \vec{M}}{\partial T} \right)_{\vec{v}, \vec{H}}] \frac{\partial T}{\partial t} + \mu_0 T \left( \frac{\partial \vec{M}}{\partial T} \right)_{\vec{v}, \vec{H}} \cdot \frac{\partial \vec{H}}{\partial t} = k_t \nabla^2 T$$

where, $C_{V,H}$ is the specific heat capacity at constant volume and magnetic field per unit mass, and $k_t$ the thermal conductivity.
The continuity equation is
\[ \nabla \cdot \dot{\rho} = 0 \]  
(10)

We follow the stability analysis as outlined in the work of Finlayson [4]. The basic state is quiescent and is given by
\[ \dot{\rho} = 0, \quad p = p_0(z), \quad T_b = T_0 - \beta z \]  
(11)

To study the stability of the system, we perturb all the variables in the form
\[ \dot{q} = q^* + \rho_0(z) + p^* \]  
(12)

where \( q^* \), \( p^* \), and \( T^* \) are perturbed variables and are assumed to be small.

Substituting Eq. (12) into Eq. (3), using Eqs. (6) and (7), and assuming that \( \beta = a(\gamma + \phi) \) as proposed by Finlayson [4], we obtain (after dropping the primes)
\[ H^* = (H_0 + M_0)H_0, \quad H^* = (H_0 + M_0)H_0, \quad H^* = (H_0 + M_0)H_0, \]  
(13)

where \((H_0, M_0, H_0)\) are \((x, y, z)\) components of perturbed magnetic field and magnetization, respectively.

Substituting Eq. (12) into Eq. (8) and linearizing, we obtain in components (after neglecting the primes)
\[ \rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \eta_0 [1 + u_0(0,0)] \nabla^2 u + \mu_0(0,0) \frac{\partial H}{\partial z} \]  
(14)

\[ \rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \eta_0 [1 + u_0(0,0)] \nabla^2 v + \mu_0(0,0) \frac{\partial H}{\partial z} \]  
(15)

\[ \rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu_0 \nabla \cdot H \frac{\partial T}{\partial z} + \mu_0 \frac{\partial H}{\partial z} + \mu_0 \frac{\partial H}{\partial z} \]  
(16)

Differentiating Eqs. (14) and (15) partially with respect to \( x \) and \( y \), respectively, and adding, we obtain
\[ \nabla^2 \dot{p} = -\rho_0 \mu_0 \frac{\partial T}{\partial z} \frac{\partial^2 \psi}{\partial z^2} + \mu_0(0,0) \theta \frac{\partial^2 \psi}{\partial z^2} \]  
(17)

where \( \nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \) is the horizontal Laplacian operator. Eliminating the pressure term from Eq. (16), using Eq. (17), we obtain
\[ \left( \rho_0 \frac{\partial}{\partial t} - \eta_0 [1 + \mu_0(0,0)] \right) \nabla^2 w = -\rho_0 \mu_0 \frac{\partial \psi}{\partial z} \frac{\partial^2 \psi}{\partial z^2} + \mu_0(0,0) \frac{\partial^2 \psi}{\partial z^2} \]  
(18)

where \( \nabla^2 = \nabla^2 + \partial^2 / \partial z^2 \) is the Laplacian operator.

As before, substituting Eq. (12) into Eq. (9) and linearizing, we obtain (after neglecting primes)
\[ \rho_0 C_0 \frac{\partial T}{\partial t} - \mu_0 K T_b \frac{\partial \psi}{\partial z} = \left( \rho_0 C_0 \frac{\partial T}{\partial t} - \mu_0 K T_b \right) w + \mu_0 K T_b \nabla^2 T \]  
(19)

Finally, Eqs. (3) and (4), after using Eqs. (12) and (13), yield
\[ \left( 1 + \frac{M_0}{H_0} \right) \nabla^2 \theta + \left( 1 + \chi \right) \frac{\partial^2 \theta}{\partial z^2} - K \frac{\partial T}{\partial z} = 0. \]  
(20)

Since the principle of exchange of stability is valid, we assume the normal mode solution in the form
\[ (w, \varphi, \psi)(x, y, z) = e^{i(x'z' + m y)} \]  
(21)

where \( l \) and \( m \) are wave numbers in the \( x \) and \( y \) directions respectively. Substituting Eq. (21) in Eqs. (18)–(20) and non-dimensionalizing the quantities in the form
\[ (x', y', z') = \left( x - \frac{a}{d} \frac{d}{C_1}, y - \frac{b}{d} \frac{d}{C_1} \right), \quad w_r = \frac{d}{C_1} w, \quad T' = \frac{d}{C_1} T, \quad \theta^* = \frac{k}{C_1 d^2} \theta, \quad \phi^* = (1 + \chi) \frac{d}{C_1 d^2} \phi \]  
(22)

we get
\[ (1 + \lambda) \left( D^2 - a^2 \right)^2 W = (R a + R m) a^2 \theta - a^2 R m \Psi \]  
(23)

\[ D^2 \theta - a^2 M z \Psi - D \Psi = 0 \]  
(24)

where \( D = d / (d z) \) is the differential operator, \( a = \sqrt{a^2 + m^2} \) is the overall horizontal wave number, \( Ra = \gamma g b^2 d^4 / C_1 \) the thermal Rayleigh number, \( Re = Ra Mz = \mu \beta \beta^2 b^4 / (1 + \gamma) \mu \) the magnetic Rayleigh number, \( A = \frac{\mu_0 M_0(0,0)}{H_0} \) the non-dimensional magnetic field dependent viscosity parameter, \( M_1 = \mu_0(0,0) \beta^2(1 + \gamma) \beta^2 \mu \) the magnetic number, \( M_2 = \frac{\mu_0 T_0(0,0)}{\mu_0 C_1(0,0)} \beta^2(1 + \gamma) \beta^2 \mu \) the non-dimensional parameter and its value for different carrier liquids turns out to be of the order of \( 10^{-6} \) and hence its effect is neglected as compared to unity.

The above equations are to be solved subject to appropriate boundary conditions. The boundary conditions considered are
\[ W = D W = \theta = \psi = 0 \text{ at } z = 0 \]  
(26)

\[ W = (1 + \lambda) D^2 W + Ma^2 \theta = D \theta + Bi \theta = D \Psi = 0 \text{ at } z = 1 \]  
(27)

where \( Ma = \sigma_1 \Delta T d j / k \) the Marangoni number and \( Bi = d h / k \) is the Biot number. The case \( Bi = 0 \) and \( Bi = \infty \) respectively, correspond to constant heat flux and isothermal conditions at the upper boundary.

3. Method of solution

Eqs. (23)–(25) together with the boundary conditions (26) and (27) constitute a Sturm–Liouville problem with the Marangoni number \( Ma \) or the Rayleigh number \( Ra \), as an eigenvalue while keeping other physical parameters fixed. To solve the resulting eigenvalue problem, Rayleigh–Ritz’s method is used. Accordingly, the variables are written in a series of basis functions as
\[ W = \sum_{i=1}^{n} A_i W_i(z), \quad \theta(z) = \sum_{i=1}^{n} C_i \theta_i(z) \text{ and } \phi(z) = \sum_{i=1}^{n} D_i \phi_i(z) \]  
(28)

where the trial functions \( W_i(z), \theta_i(z) \) and \( \phi_i(z) \) will be generally chosen in such a way that they satisfy the respective boundary conditions, and \( A_i, C_i \) and \( D_i \) are constants. Substituting Eq. (28) into Eqs. (23)–(25), multiplying the resulting momentum Eq. (18) by \( W_i(z) \), energy Eq. (19) by \( \theta_i(z) \) and magnetic potential Eq. (20) by \( \phi_i(z) \); performing the integration by parts with respect to \( z \) between \( z = 0 \) and \( z = 1 \) and using the boundary conditions (26) and (27), we obtain the following system of linear homogeneous
algebraic equations:
\[ C_j A_j + D_j C_j + E_j D_j = 0 \]  
\[ F_j A_j + G_j C_j = 0 \]  
\[ H_j C_j + I_j D_j = 0 \]  
(29), (30), (31)

The coefficients \( C_j - I_j \) involve the inner products of the basis functions and are given by
\[ C_j = (1 + \Lambda)\left[ (\in^T W_i D W_i) + 2a^2 (W_i D W_i) + \alpha^2 (W_i D W_i) \right] \]

\[ D_j = -a^2 (Ra + Rm) (\Theta_i W_i) + a^2 Ma D W_i (1) \Theta_i (1) \]

\[ E_j = a^2 Rm (W_i D \Phi_i) \]

\[ F_j = -\langle \Theta_i W_i \rangle \]

\[ G_j = \langle D \Theta_i D \Theta_i \rangle + a^2 \langle \Theta_i \Theta_i \rangle + Bi \Theta_i (1) \Theta_i (1) \]

\[ H_j = \langle \Phi_i D \Theta_i \rangle \]

\[ I_j = \langle D \Phi_i D \Phi_i \rangle + a^2 M_M^z \langle \Phi_i \Phi_i \rangle \]

where the inner product is defined as \( \langle \cdot \cdot \cdot \rangle = \int_0^1 (\cdot \cdot \cdot) \, dz \). The above set of homogeneous algebraic equations can have a non-trivial solution if and only if
\[ \begin{vmatrix} C_j & D_j & E_j \\ F_j & G_j & 0 \\ 0 & H_j & I_j \end{vmatrix} = 0 \]

(32)

The eigenvalue has to be extracted from the characteristic Eq.(32). We select the trial functions as
\[ W_i = z^2 (1-z) T_{\alpha}^{(i)} \; \Theta_i = z (1-z) T_{\alpha}^{(i)} \; \Phi_i = z^2 (1-2z/3) T_{\alpha}^{(i)} \]

(33)

where \( T_{\alpha}^{(i)} \)’s are the Chebyshov polynomials of the second kind, such that \( W_i, \Theta_i \) and \( \Phi_i \) satisfy the corresponding boundary conditions except, \( (1 + \Lambda) D W_i + Ma a^2 \Theta = 0 = D \Theta + Bi \Theta \) at \( z = 1 \) but the residuals from the equations are included as residuals from the differential equations.

4. Numerical results and discussion

It may be noted that Eq.(32) leads to the characteristic equation giving the Marangoni number \( Ma \) or the Rayleigh number \( Ra \) as a function of the wavenumber \( a \), the parameters \( Ra, M_M, L_1, A, A, B, M_2 \) and \( \Lambda \). The inner products involved in the equation are evaluated analytically in order to avoid errors in the numerical integration. Computations reveal that the convergence in finding \( Ma \) or \( Ra \) crucially depends on the value of MFD viscosity parameter \( \Lambda \). The results presented here are for \( i = j = 6 \) the order at which the convergence is achieved, in general. In order to validate the numerical solution procedure used, first the critical values \( (Ra, Ma, q_1) \) obtained from the present study under the limiting conditions are compared with the previously published results of Davis [25] in Table 1. The results tabulated in Table 1 for different values of heat transfer coefficient \( Bi \) (i.e. Biot number) are for \( \Lambda = 0 \) and \( Rm = 0 \) (i.e., classical Bénard–Marangoni convection for ordinary viscous fluid). From the table it is evident that there is an excellent agreement between the results of the present study and the previously published ones. This verifies the applicability and accuracy of the method used in solving the problem.

![Fig. 1](image-url)

(a) Locus of critical Marangoni number \( Ma \) and Rayleigh number \( Ra \) with \( Rm=0 \) and \( \Lambda=0 \).

(b) Critical wave number \( q_1 \) as a function of \( Ra \) for different values of \( \Lambda \) for \( Bi=2, M_1=2 \) and \( M_2=1 \).
We now look into the solution of the complete problem, which involves the effect of all the parameters $R_a$, $R_m$, $B_i$, $L$, $M_1$, and $M_3$ on the criterion for the onset of convection. The salient characteristics of these parameters are exhibited graphically in Figs. 1–7 and also in Table 2. These figures exhibit a tight coupling between the buoyancy, magnetization and surface tension forces. Fig. 1(a) shows the locus of the critical Marangoni number $M_{ac}$ and the Rayleigh number $R_{ac}$ for different values of MFD viscosity parameter $L$ for $B_i=2$, $M_1=2$ and $M_3=1$. From the figure, it is obvious that there is a strong coupling between the critical Rayleigh and the Marangoni numbers, and an increase in the Rayleigh number has a destabilizing effect on the system. Thus, when the buoyancy force is predominant, the surface tension force becomes negligible and vice-versa. From Fig. 1(a), it is seen that the critical Rayleigh and Marangoni numbers increase with an increase in the MFD viscosity parameter and thus it has a stabilizing effect on the system. That is, the effect of increasing $A$ is to delay the onset of Bénard–Marangoni ferroconvection. The variation in $a_c$ as a function of $Ra$ is elucidated in Fig. 1(b) for different values of $A$ with $Bi=2$, $M_1=2$ and $M_3=1$. It may be noted that the curves of different $A$ cross over each other with an increase in the value of $Ra$. That is, an increase in the value of $A$ increases marginally the critical wave number $a_c$ up to some value of $Ra$, depending on the value of $A$, and an opposite trend prevails with further increase in the value of $Ra$.

The plots in Fig. 2(a) represents the locus of critical Marangoni number $M_{ac}$ and Rayleigh number $R_{ac}$ for different values of $M_1$ for $A=0.2$, $Bi=2$ and $M_3=1$. (b) Critical wave number $a_c$ as a function of $Ra$ for different values of $M_1$ for $A=0.2$, $Bi=2$ and $M_3=1$.

Fig. 3. (a) Locus of critical Marangoni number $M_{ac}$ and Rayleigh number $R_{ac}$ for different values of $M_1$ for $A=0.2$, $Bi=2$ and $M_3=1$. (b) Critical wave number $a_c$ as a function of $Ra$ for different values of $M_1$ for $A=0.2$, $Bi=2$ and $M_3=1$. 
more stable. Fig. 2(b) represents the corresponding critical wave number and it indicates that increase in the value of \( \text{Bi} \) is to increase \( a_c \) and thus its effect is to reduce the size of convection cells. It is also seen that the critical wave number passes through a minimum with increasing \( \text{Ra} \).

Fig. 3(a) presents the locus of the critical values of \( R_{ac} \) and \( M_{ac} \) for various values of magnetic number \( M_3 \) for \( \lambda = 0.2, \text{Bi}=2 \) and \( M_1=2 \). The curve of \( M_1=0 \) corresponds to the case when only the buoyancy force is in effect and it lies above all other curves of different \( M_1 \). This indicates that increasing \( M_1 \) is to make the system more unstable due to increase in the destabilizing magnetic force. Besides, the curves of different \( M_1 \) become closer as the value of \( M_1 \) increases. Although the critical wave number \( a_c \) remains invariant for different values of \( M_1 \) at lower values of \( Ra \) it increases with further increase in the value of \( Ra \) (see Fig. 3(b)). Further, the deviation in the critical wave number amongst different values of \( M_1 \) increases with increasing \( M_1 \) as well as \( Ra \).

Fig. 4(a) presents the critical Marangoni number \( M_{ac} \) as a function of critical Rayleigh number \( R_{ac} \) for several values of nonlinearity of magnetization parameter \( M_3 \) for \( \lambda = 0.2, \text{Bi}=2 \) and \( M_1=2 \). It can be seen that an increase in \( M_3 \) is to decrease \( R_{ac} \) and \( M_{ac} \) but only marginally and thus it has a destabilizing effect on the stability of the system. This may be due to the fact that the application of magnetic field makes the ferrofluid to acquire larger magnetization which in turn interacts with the imposed magnetic field and releases more energy to drive the flow faster. Hence, the system becomes unstable with a smaller temperature gradient as the value of \( M_3 \) increases. Alternatively, a higher value of \( M_3 \) would arise either due to a larger pyromagnetic coefficient or larger temperature gradient. Both these factors are conducive for generating a larger gradient in the Kelvin body force field, possibly promoting the instability.

The variation of critical wave number \( a_c \) as a function of Rayleigh number \( Ra \) is shown in Fig. 4(b) for different values of \( M_3 \). From the figure, we note that an increase in \( M_3 \) is to increase \( a_c \) and hence its effect is to decrease the dimension of convection cells. From the figure it is also seen that the critical wave number
decreases initially with increasing $Ra$ but eventually increases with further increase in the value of $Ra$.

Figs. 5–7 show the critical values of $Ma_c$ (pure Marangoni ferroconvection) and $Ra_c$ (pure Bénard ferroconvection) as well as corresponding $a_c$ for different values of $Bi$, $Rm$ and $M_3$, respectively, as a function of MFD viscosity parameter $\Lambda$. From the figures, it is seen that $Ma_c < Ra_c$ and the effect of increasing $\Lambda$ is to delay the onset of Bénard/Marangoni ferroconvection. Further, increase in $Bi$ (Fig. 5(a)) and decrease in $Rm$ (Fig. 6(a)) and $M_3$ (Fig. 7(a)) is to increase the critical Rayleigh/Marangoni number and hence has a stabilizing effect on the system. Moreover, increase in $Bi$ (Fig. 5(b)), $Rm$ (Fig. 6) and $M_3$ (Fig. 7(b)) is to decrease the width of convection cells. The critical wave numbers $a_c$ for Bénard ferroconvection are always found to be higher than those of pure Marangoni ferroconvection (see Figs. 5–7(b)). Further inspection of these figures reveals that the variation in $a_c$ with $\Lambda$ is insignificant but for different values of $M_3$ it decreases monotonically with $\Lambda$.

The tight coupling between buoyancy, surface tension and magnetic forces is exhibited quantitatively by tabulating the values of triplets $(Ra_c, Ma_c, Rmc)$ for different values of $\Lambda$ and $M_3$ with $Bi=2$ in Table 2. It is observed that increase in one of these decreases the other and vice-versa. As $M_3$ increases, $Rmc$ decreases and the results reduce to that of classical Bénard–Marangoni problem for ordinary viscous fluids as $M_3 \to \infty$. That is, $Rmc=Rmc$ as $M_3 \to \infty$.

5. Conclusions

The effect of MFD viscosity on the criterion for the onset of coupled Bénard–Marangoni convection in a ferrofluid layer is investigated since the viscosity of the magnetic fluid varies with an applied magnetic field. The lower rigid surface of the ferrofluid layer is heated from below, while a general thermal condition is used at the upper free surface subjected to a surface tension decreasing with temperature. The resulting eigenvalue problem is solved numerically by employing the Rayleigh–Ritz technique with either Rayleigh number ($Ra$) or Marangoni number ($Ma$) as
the eigenvalue. The effect of magnetic field dependent viscosity measured through the parameter $\lambda$ on the physical parameters of importance $Ra$ as well as $Ma$ is analyzed in detail.

The following conclusions can be drawn from the present study:

(i) The effect of increase in the value of magnetic field dependent viscosity parameter $\lambda$ is to increase the value of critical stability parameters $Ra_c$ or $Ma_c$ and hence its effect is to delay the onset of Bénard–Marangoni ferroconvection.

(ii) Increase in the value of Biot number $Bi$ is to delay the onset of Bénard–Marangoni ferroconvection, while increase in the value of magnetic Rayleigh number $R_m$ and nonlinearity of fluid magnetization parameter $M_3$ is to advance the onset of Bénard–Marangoni ferroconvection.

(iii) The buoyancy and surface tension forces complement with each other and it is always found that $Ma_c < Ra_c$; a result in accordance with ordinary viscous fluids.

(iv) The effect of increase in $Bi$ and $\lambda$ as well as decrease in $M_1$ and $M_3$ values is to decrease the dimension of the convection cells.

(v) As $M_3 \rightarrow \infty$, the results reduce to that of the Bénard–Marangoni problem for ordinary viscous fluids.

Acknowledgements

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References


Table 2

The critical instability parameters $Ra_c$ and $R_{mc}$ for different values of $\lambda$ and $Ma$ when $Bi=2$.

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<th>$\lambda$</th>
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