Penetrative Brinkman convection in an anisotropic porous layer saturated by a nanofluid

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Abstract The onset of penetrative Brinkman convection in a nanofluid saturated anisotropic porous layer is investigated via uniform internal heating for rigid-rigid, free-free, and lower-rigid and upper-free boundaries. The flux of volume fraction of nanoparticles is taken to be zero on the isothermal boundaries and the eigenvalue problem is solved using the Galerkin method. The numerical computations carried out indicated the validity of principle of exchange of stability for all types of velocity boundary conditions. The effect of heat source strength, mechanical anisotropy parameter, modified diffusivity ratio, nanoparticle concentration Darcy-Rayleigh number and Lewis number is to hasten, while the Darcy number and thermal anisotropy parameter are to delay the onset of convection. In contrast to the regular fluid saturating a Darcy porous medium, the onset of convection for nanofluids is found to be influenced even when the ratio of mechanical anisotropy parameter to thermal anisotropy parameter is unity.

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1. Introduction

Fluids with nanoparticles suspended stably in them are called nanofluids, a term proposed by Choi [1]. These fluids are considered to be the next-generation heat transfer fluids as they offer exciting new possibilities to enhance heat transfer perfor-
suspensions. Kim et al. [10] investigated the convective instability driven by thermal gradient and heat transfer characteristics of nanofluids by introducing a new factor which quantifies the effect of nanoparticles’ addition on a base fluid.

The Darcy–Bénard problem with nanofluids has also attracted equal importance in the literature because of its importance in many fields of modern science, engineering and technology, chemical and nuclear industries and biomechanics [2,3,11–13]. Following the formalism of Nield and Kuznetsov [11], several studies were undertaken subsequently to investigate various additional effects on the problem by the same authors and others considering the volume fraction of nanoparticles is constant at the boundaries [12–18]. Recently, Nield and Kuznetsov [19] pointed out that this type of boundary condition on volume fraction of nanoparticles is physically not realistic as it is difficult to control the nanoparticle volume fraction on the boundaries. They suggested the normal flux of volume fraction of nanoparticles is zero on the boundaries as an alternative boundary condition which is physically more realistic. Under the circumstances, it is desirable to investigate convective instability problems by utilizing these boundary conditions to get meaningful insight into the problems. The details can be found in the book by Nield and Bejan [20].

The penetration of fluid from a thermally convectively unstable region into a neighboring stably stratified region due to a different temperature environment is known as penetrative convection. There are several ways to describe penetrative convection, and one of the models widely used to describe it is via internal heating. There exists an extensive literature on this subject because of its importance in geophysics, astrophysics, thermal ignition, fire and combustion modeling, miniaturization of electronic components, etc (for details see [21–26]). Straughan [27] surveyed the areas in which penetrative convection occurs and he has also studied various solutions to the mathematical models that have been used to describe the same. The problem of penetrative convection in a Newtonian fluid-saturated porous layer has also received considerable attention in the recent past [28–31]. The effect of boundary and internal heat source on the onset of Darcy-Brinkman convection in a porous layer saturated by a nanofluid is studied by Yadav et al. [12] while the onset of convection solely due to internal heating in a Darcy porous medium saturated with nanofluid is considered by Nield and Kuznetsov [32].

It is known that high porosity porous materials (for example, foameitals) are used in many practical applications such as heat exchangers, chemical reactors and fluid filters to mention a few. Hence, high porosity materials are of much current interest in many technological problems. They are typically man-made and are important in the design of heat transfer devices. In dealing such highly porous materials the use of the higher order Darcy-Brinkman equation becomes more appropriate to model the fluid flow, as opposed to the more
commonly used Darcy’s law [33]. Anisotropy is generally a consequence of preferential orientation or asymmetric geometry of porous matrix or fibers and is in fact encountered in numerous systems in industry and nature. Furthermore, anisotropy can also be a characteristic of artificial porous materials such as pelleting used in chemical engineering process; fiber material used in insulating purpose and packed beds used in the storage of heat energy. Anisotropy effects in general have been reviewed by McKibbin [34,35] and Storesletten [36,37]. Copious literature available on the non-Darcy–Benard convection in isotropic and anisotropic porous media is amply documented in the book by Nield and Bejan [20]. Chand et al. [17] studied the effects of variable gravity on thermal instability in a horizontal layer of a nanofluid saturating an anisotropic Darcy porous medium.

The intent of the present paper is to study the onset of penetrative convection in a horizontal layer of an anisotropic Brinkman porous medium saturated by a nanofluid via internal heating and heated uniformly from below. Such a study finds relevance in many applications particularly in manufacturing processes in industry. It is imperative to note that the internal heat generation changes the temperature distribution significantly in the nanofluid which eventually alters the particle deposition rate in nuclear reactors, electronic chips and semiconductor wafers. In investigating the problem, physically more realistic boundary condition on the volume fraction of nanoparticles is used. That is, the normal flux of volume fraction of nanoparticles is taken to be zero at the boundaries. Due to the presence of internal heating and changed boundary conditions, the resulting generalized eigenvalue problem is found to be not amenable for analytical treatment. Hence, the critical stability parameters are extracted numerically using the Galerkin method for different types of velocity boundary conditions. The study undertaken is more general in the sense that the results for the Darcy porous medium as well as for a fluid layer can be delineated as particular cases from the present study.

2. Mathematical formulation

The physical configuration is as shown in Fig. 1. We consider penetrative convection via uniform internal heating in a system consisting of a horizontal layer of an incompressible nanofluid-saturated Brinkman anisotropic porous layer of thickness d heated uniformly from below. To account for Brownian diffusion and thermophoretic effects, Buongiorno’s nanofluid model has been utilized. A Cartesian coordinate system (x, y, z) is chosen such that the origin is at the bottom of the porous layer and the gravity is acting in the negative vertical z-direction. The lower and upper impermeable boundaries are maintained at constant but different temperatures T₀ and T₁ (< T₀), respectively. The fluid and solid phases are assumed to be in local thermal equilibrium. The governing equations under the Boussinesq approximation are the conservation of mass, momentum, thermal energy and nanoparticles, and they are respectively given by [20,38]

$$\nabla \cdot \vec{q} = 0$$  \hspace{1cm} (1)

$$\frac{\rho_p}{\varepsilon} \frac{\partial \vec{q}}{\partial t} = -\nabla p + \vec{u} \nabla^2 \vec{q} - \mu (K^{-1} \cdot \vec{q}) + \left[ \frac{\phi_p}{\varepsilon} + (1 - \phi_p) \rho_f \left( 1 - \beta (T - T_0) \right) \right] \vec{g}$$  \hspace{1cm} (2)

$$(\rho \varepsilon) \frac{\partial T}{\partial t} + (\rho \varepsilon) \left( \vec{q} \cdot \nabla \right) T = \nabla \cdot (k_m \cdot \nabla T)$$  \hspace{1cm} (3)

$$\frac{\partial \phi}{\partial t} + \frac{1}{\varepsilon} (\vec{q} \cdot \nabla) \phi = \nabla \cdot (D_b \nabla \phi + \frac{D_f}{\varepsilon} \nabla T) + Q_b$$  \hspace{1cm} (4)

where $\vec{q} = (u, v, w)$ the velocity vector, $\rho$ the pressure, $\rho_p$ the density of nanoparticles, $\rho_f$ the density of base fluid, $\vec{g}$ the acceleration due to gravity, $\mu$ the dynamic viscosity of the fluid, $\beta$ the effective viscosity, $K$ the permeability tensor of the porous medium, $\varepsilon$ the porosity of the porous medium, $T$ the temperature of the nanofluid, $\phi$ the nanoparticle volume fraction, $\beta$ the coefficient of thermal expansion, $c$ the specific heat at constant pressure, $k_m$ the thermal conductivity tensor, $Q_b$ is the uniformly distributed volumetric internal heat generation source, $D_b$ the Brownian diffusion coefficient and $D_f$ the thermophoretic diffusion coefficient. In Eqs. (3) and (4), the terms within the square brackets represent the flux of nanoparticles due to Brownian diffusion and thermophoresis. The permeability and thermal conductivity tensors are defined as

$$k_{\sim}^1 = K_H^1 \left( \vec{i} \cdot \vec{j} \right) + k_{\sim}^{1,1} \vec{k} \vec{k}$$  \hspace{1cm} (5)

$$k_{\sim} = k_{mil} \left( \vec{i} \cdot \vec{j} \right) + k_{mil} \vec{k} \vec{k}$$  \hspace{1cm} (6)

where $K_H$ is the permeability and $k_{mil}$ is the thermal conductivity in the horizontal i and j directions, while $K_V$ and $k_{mil}$ are the corresponding values in the vertical k direction. It may be noted that horizontal mechanical and thermal isotropy has been assumed. To non-dimensionalize the governing Eqs. (1)-(4), the variables are scaled as follows:

$$(x', y', z') = (x, y, z)/d, \quad \vec{q} = (d/\kappa_c) \vec{q}, \quad t' = (\kappa_v/a d^2) t, \quad p' = (K_v/m \kappa_v) p, \quad \phi' = (\phi - \phi_0)/\phi_0, \quad T' = (T - T_0)/\Delta T$$  \hspace{1cm} (7)

Figure 1 Physical configuration.
Here, \( R_t = dK_b \beta \rho A T_g / \mu k_v \) is the thermal Darcy-Rayleigh number, \( Pr = \mu c_v^2 / \rho c_v \) is the Darcy-Prandtl number, \( R_a = K_d (\rho_c - \rho) \phi_b / \mu_c \) is the concentration Darcy-Rayleigh number, \( Le = k_v / \mu_b \) is the Lewis number, \( N_a = D / \Delta T / Du T_0 \) is the modified diffusivity ratio, \( N_r = (\rho_c) \phi_b / (\rho c_v) \) is the modified particle density increment, \( Du = K_b \bar{\mu} / \mu_f \) is the Darcy number, \( \xi = K_m / \xi \) is the mechanical anisotropy parameter, \( n = k_{anl} / k_{nor} \) is the thermal anisotropy parameter, \( N_s = d^2 q_e / 2 \kappa_3 (\rho c_v) A T \) is the dimensionless heat source strength, \( M = (\rho c_v) n_c \mu_c / \rho c_v \) is the heat capacity ratio, \( \nabla^2 h = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \) is the horizontal Laplacian operator and \( \nabla^2 \phi = \nabla^2 h + \partial^2 / \partial z^2 \). In obtaining Eq. (9), a term proportional to the product of \( \phi \) and \( T \) is neglected in the spirit of the Oberbeck–Boussinesq approximation and this is valid in the case of small temperature gradients in a dilute suspension of nanoparticles following Nield and Kuznetsov [11].

The isothermal boundaries are considered to be either rigid or free. However, the consideration of boundary conditions on the volume fraction of nanoparticles is a matter of concern. Recently, Nield and Kuznetsov [19] looked into this ambiguity and suggested the normal flux of nanoparticle volume fraction is zero on the boundaries as an alternative boundary condition which is physically more realistic.

Thus the proposed boundary conditions are
\[
w = 0, \frac{\partial w}{\partial z} = 0 (\text{rigid}) \text{ or } \frac{\partial w}{\partial z} = 0 (\text{free}), T = 0, \frac{\partial T}{\partial z} + N_s \frac{\partial T}{\partial z} = 0 \text{ at } z = 0 \tag{12}
\]
\[
w = 0, \frac{\partial w}{\partial z} = 0 (\text{rigid}) \text{ or } \frac{\partial w}{\partial z} = 0 (\text{free}), T = -1, \frac{\partial \phi}{\partial z} + N_s \frac{\partial T}{\partial z} = 0 \text{ at } z = 1. \tag{13}
\]

In the quiescent basic state, the temperature and volume fraction of nanoparticles vary only in the vertical \( z \)-direction and satisfy the following equations:
\[
\frac{d^2 T_b}{dz^2} + N_s \frac{dT_b}{dz} + \frac{N_s N_b}{Le} \left( \frac{dT_b}{dz} \right)^2 + 2 N_s = 0 \tag{14}
\]
\[
N_s \frac{d^2 T_b}{dz^2} + \frac{d^2 \phi_b}{dz^2} = 0. \tag{15}
\]

The above equations are solved subject to the boundary conditions
\[
T_b = 0, \frac{d\phi}{dz} + N_s \frac{dT_b}{dz} = 0 \text{ at } z = 0 \tag{16a, b}
\]
\[
T_b = -1, \frac{d\phi}{dz} + N_s \frac{dT_b}{dz} = 0 \text{ at } z = 1. \tag{17a, b}
\]

Integrating Eq. (15) once with respect to \( z \) and using the boundary condition (16b), we get
\[
\frac{d\phi_b}{dz} + N_s \frac{dT_b}{dz} = 0. \tag{18}
\]
Using Eq. (18) in Eq. (14), we obtain
\[
\frac{d^2 T_b}{dz^2} = -2 N_s. \tag{19}
\]

On integrating (19) with respect to \( z \) twice and using the boundary conditions (16a) and (17a), we get
\[
T_b = -z + N_s z (1 - z). \tag{20}
\]

Now using Eq. (20) in Eq. (18) and solving, we get
\[
\phi_b = \phi_0 - N_s \{ -z + N_s z (1 - z) \} \tag{21}
\]

where \( \phi_0 \) is the reference value of nanoparticle volume fraction. Thus we note that the effect of internal heating is to alter the basic temperature and volume fraction of nanoparticles distributions from linear to nonlinear with respect to porous layer height.

### 3. Linear stability analysis

The basic state is perturbed in the form
\[
\tilde{q} = \tilde{q}, \quad T = T_b(z) + \tilde{T}, \quad \phi = \phi_b(z) + \phi'. \tag{22}
\]
where \( \tilde{q}, \tilde{T} \) and \( \phi' \) are the perturbed quantities over their equilibrium counterparts and assumed to be small.

Substituting Eq. (22) into Eqs. (8)–(11) and linearizing, we obtain the stability equations in the form
\[
\frac{1}{Pr} \frac{\partial \tilde{q}}{\partial z} (\nabla^2 w') = - \left( \frac{1}{\xi} \frac{\partial^2 w'}{\partial x^2} + \nabla^2 w' \right) + \nabla^2 \nabla^2 w' - R_e \nabla^2 \phi' + R_s \nabla^2 T' \tag{23}
\]
\[
\frac{\partial \tilde{T}}{\partial z} = -w' f(z) + \frac{\partial T'}{\partial z} + \eta \nabla^2 T' + N_s N_b f(z) \frac{\partial T'}{\partial z} + N_s f(z) \frac{\partial \phi'}{\partial z} \tag{24}
\]
\[
\frac{\partial \phi'}{\partial z} = N_s f(z) w' + N_s \nabla^2 T' + \frac{1}{Le} \nabla^2 \phi'. \tag{25}
\]

where
\[
f(z) = \{ N_s (1 - 2z) - 1 \} \tag{26}
\]

The boundary conditions upon using Eq. (22) become
\[
w' = 0, \quad \frac{\partial \tilde{q}}{\partial z} = 0 (\text{rigid}) \text{ or } \frac{\partial^2 w'}{\partial z^2} = 0 (\text{free}), \tag{27}
\]
\[
T' = 0, \quad \frac{\partial \tilde{T}}{\partial z} = \frac{\partial T'}{\partial z} = 0 \text{ at } z = 0 \tag{28}
\]

The normal mode expansion of the dependent variables is assumed in the form
\[
[w', T', \phi'] = [W(z), \Theta(z), \Phi(z)] \text{e}^{i(\omega t + l x + m y)} \tag{29}
\]

where \( \omega (= \omega_x + i \omega_y) \) is the growth rate, \( l \) and \( m \) are the wave numbers in the \( x \) and \( y \)-directions, respectively. Eq. (29) is substituted into Eqs. (23)–(25) and in the boundary conditions (27) and (28) to yield the following system of stability equations and the boundary conditions:
\[
Du(D^2 - \alpha^2) W - \left( \frac{1}{\xi} \frac{1}{D^2 - \alpha^2} \right) W - \alpha R_e \Theta + \alpha R_s \Phi = \omega \frac{1}{Pr} (D^2 - \alpha^2) W \tag{30}
\]
\[- W f(z) + \frac{1}{Le} (D^2 - \alpha^2) \Theta + \frac{1}{Le} (D^2 - \alpha^2) \Phi = M \alpha \Theta \tag{31}
\]
\[N_s f(z) W + N_s \frac{1}{Le} (D^2 - \alpha^2) \Theta + \frac{1}{Le} (D^2 - \alpha^2) \Phi = \omega \Phi. \tag{32}
\]
where \( D = d/dz \).

The boundary conditions are
\[
W = 0, \quad DW = 0 (\text{rigid}) \quad \text{or} \quad D^2W = 0 (\text{free}), \quad \Theta = 0, \quad D\Phi + N_\Phi \Theta = 0 \quad \text{at} \quad z = 0
\]
\[
W = 0, \quad DW = 0 (\text{rigid}) \quad \text{or} \quad D^2W = 0 (\text{free}), \quad \Theta = 0, \quad D\Phi + N_\Phi \Theta = 0 \quad \text{at} \quad z = 1.
\]

(33)

(34)

4. Numerical solution

Eqs. (30)-(32) together with the boundary conditions (33) and (34) constitute a linear eigenvalue problem with variable coefficient for the growth rate \( \omega \) of the system. The resulting eigenvalue problem is solved numerically by the Galerkin method to get more accurate results. Accordingly, the variables are written in a series of basis functions as
\[
W(z) = \sum_{i=1}^{N} A_i W_i(z), \quad \Theta(z) = \sum_{i=1}^{N} B_i \Theta_i(z), \quad \Phi(z) = \sum_{i=1}^{N} C_i \Phi_i(z)
\]
(35)

where \( A_i, B_i, \) and \( C_i \) are unknown coefficients. Multiplying Eq. (30) by \( W_i(z) \), Eq. (31) by \( \Theta_i(z) \) and Eq. (32) by \( \Phi_i(z) \); performing the integration by parts with respect to \( z \) between \( z = 0 \) and \( 1 \), and using the boundary conditions, we obtain the following system of algebraic equations:
\[
A_i C_i + B_i D_i + C_i E_i = \omega A_i F_i
\]
\[
A_i G_i + B_i H_i + C_i J_i = \omega B_i J_i
\]
\[
A_i K_i + B_i L_i + C_i M_i = \omega C_i N_i
\]
(36)

The coefficients \( C_i - N_i \) involve the inner products of the basis functions and are given by
\[
C_i = D \Delta \{ \langle DW_i DW_j \rangle + 2a \langle DW_i DW_j \rangle + a^2 \langle W_i W_j \rangle \}
\]
\[
+ (1/\zeta) \langle DW_i DW_j \rangle + a^2 \langle W_i W_j \rangle,
\]
\[
D_i = -a R_i \langle W_i \Theta_i \rangle, \quad E_i = a^2 R_i \langle W_i \Phi_i \rangle,
\]
\[
F_i = -(1/Pr \zeta) \{ \langle DW_i DW_j \rangle + a^2 \langle W_i W_j \rangle \},
\]
\[
G_i = \langle f(z) \Theta_i W_i \rangle,
\]
\[
H_i = -\langle (D\Theta_i D\Theta_i) - \eta a^2 \langle \Theta_i \Theta_i \rangle \rangle + N_A N_B \langle f(z) \Theta_i D\Theta_i \rangle,
\]
\[
J_i = N_A \langle f(z) \Theta_i D\Phi_i \rangle, \quad J_j = M_i \langle \Theta_i \Theta_i \rangle, \quad K_i = N_A \langle f(z) \Phi_i W_i \rangle.
\]
(37)

\[
L_i = -N_A \langle \{ (D\Phi_i D\Theta_i) + a^2 \langle \Phi_i \Phi_i \rangle \}, \quad M_i = -1 \langle \{ (D\Phi_i D\Phi_i) + a^2 \langle \Phi_i \Phi_i \rangle \}, \quad N_i = \langle \Phi_i \Phi_i \rangle.
\]

The system of equations given by Eq. (36) is a generalized eigenvalue problem which can be written in the form
\[
AX = \omega BX
\]
(38)

We note that \( A \) and \( B \) are real matrices of order \( N \times N \) and \( X \) is the eigenvector. The basis functions are represented by the power series satisfying the respective boundary conditions
\[
W_i(z) = z^{i-1} - 2z^{i-2} + z'(\text{free-free}),
\]
\[
W_i(z) = z^{i-1} - 2z^{i-2} + z^i (\text{rigid-rigid}),
\]
\[
W_i(z) = 2z^{i-1} - 5z^{i-2} + 3z^{i-3} (\text{rigid-free}),
\]
\[
\Theta_i(z) = z^{i-1}, \quad \Phi_i(z) = N_A (z^{i-1} - z^i).
\]

The results for \( i = 1 \) correspond to the single term Galerkin method and in that case an explicit formula for the Rayleigh number can be obtained. Nonetheless, such an expression for the Rayleigh number is not feasible for higher values of \( i \) and \( j \), and hence the generalized eigenvalue problem given by Eq. (37) is solved numerically by using the subroutine GVLRE of the IMSL library. The complex eigenvalue \( \omega \) is determined when the other parameters are specified. Then one of the parameters, say \( R_0 \), is varied for a fixed value of wave number \( a \) until the real part of \( \omega \) vanishes. The zero crossing of \( \omega \) is achieved by Newton’s method for a fixed point determination. The corresponding values of \( R_0 \) and \( a \) are the critical conditions for neutral stability. Then the critical thermal Darcy-Rayleigh number with respect to the wave number is calculated using the golden section search method. The imaginary part of \( \omega \) indicates whether the instability onsets into steady convection or into growing oscillations. For a range of parametric values chosen and for different velocity boundary conditions considered, it is observed that \( \omega \) is always zero indicating the non-occurrence of oscillatory convection. In other words, the principle of exchange of stability is valid for the problem considered. This is expected from the physical grounds as well as there is no mechanism to set up oscillatory motions and also \( R_0 \) can take only positive values. The critical stability parameters \( (R_0, a) \) are computed for different values of physical parameters and the results are exhibited graphically in Figs. 2-9. The convergence of the results is achieved by considering eight terms in the Galerkin expansion.

5. Results and discussion

The onset of penetrative Brinkman convection via uniform internal heating is investigated in a horizontal layer of an anisotropic porous layer saturated by a nanofluid. The boundaries are considered to be rigid or free, isothermal and the normal flux of volume fraction of nanoparticles is assumed to be zero on the boundaries. The resulting generalized eigenvalue problem is solved numerically using the Galerkin method. Three different types of velocity boundary conditions, namely both boundaries free (free-free), both boundaries rigid (rigid-rigid) and lower rigid-upper free (rigid-free) are considered. The parametric values vary with the base fluid and nanoparticles chosen. The ratio of density of the nanoparticles to that of a base fluid for Cu (copper) and Ag (silver) is 8.96 and 10.5, respectively. The ratio of heat capacity based on the volume fraction of nanoparticles to that of a base fluid is 0.83 for Cu and 0.59 for Ag. Following Tzou [6,7], Buongiorno [8] and Nield and Kuznetsov [19], we have taken the values of concentration Darcy-Rayleigh number \( R_0 \) to be positive always and of the order \( 10^{-3} \)–\( 10^{-5} \) and Lewis number \( Le \) is taken...
in the order of 1–10. The value of modified diffusivity ratio \( N_A \) is not more than 10. The numerical computations carried out for a wide range of parametric values revealed that the instability always sets in via stationary convection; a result observed by Nield and Kuznetsov [19] as well for the case of isotropic Darcy-porous medium in the absence of internal heat generation. This fact also reiterates that internal heating and anisotropy effects do not support oscillatory convection as observed in regular fluids.

To check the accuracy of the numerical computations carried out, the regular critical thermal Rayleigh number \( R_c \) and the corresponding critical wave number \( a_c \) are computed for different values of \( N_S \) for rigid-rigid and rigid-free boundaries for a regular fluid (\( N_A = 0 \) and \( N_B = 0 \)) in the absence of a porous medium (i.e. for a clear fluid layer). The computed values are tabulated in Table 1 and compared with Char and Chiang [38]. It is noted that the results are in good agreement. Further, the values of \( R_c \) and the corresponding \( a_c \) for different values of \( Da \) are computed for the case of rigid-rigid boundaries and compared with Guo and Kaloni [39] in Table 2. Again, it is seen that our results are in excellent agreement. These comparisons verify the accuracy of the numerical method used in the present study.

The variation of critical Darcy-Rayleigh number \( R_{tc} \) and the corresponding critical wave number \( a_c \) as a function of dimensionless internal heat source strength \( N_S \) is presented in Figs. 2–4 for two values of \( N_A \) (with \( R_n = 2, \ Le = 10, \ N_B = 0.01, \ Da = 0.8, \ \xi = 0.8, \ \eta = 0.6 \)). In these figures the results obtained for three different types of velocity boundary conditions, namely...
rigid-rigid, rigid-free and free-free are compared. It is noted that, except for a quantitative change, the qualitative behavior of the stability of the system remains the same for different boundaries. As it is intuitively to anticipate from a physical point of view, the rigid-rigid boundaries lead to a substantial stabilizing effect compared to rigid-free and free-free boundaries. The effect of increasing $NS$ is to increase the heat supply to the system which eventually leads to decrease in the value of $R_{tc}$. The size of the convection cells gets reduced for all types of boundary conditions with the increase in the value of $NS$ (Figs. 2b, 3b and 4b). Although thermophoresis and Brownian motion are responsible for the motion of nanoparticles in the base fluid, it is observed that thermophoresis is more dominating effect to initiate the diffusion of nanoparticles and hence increase in the value of $N_A$ is to hasten the onset of convection (Fig. 2a). Increase in the value of $R_m$ is to advance the onset of convection because increase in the density of nanoparticles is to enhance the heat transfer (Fig. 3a). The effect of increasing $Le$ hastens the onset of convection as thermal diffusion is dominated over Brownian diffusion (Fig. 4a). With increasing heat source strength, the curves of $R_m$ for rigid-free and free-free boundaries get closer. For all the boundary conditions, the size of convection cells decreases with increasing $NS$ while it increases with increasing $N_A$ (Fig. 3a). Besides, $(a_c)_{rigid-rigid} > (a_c)_{rigid-free} > (a_c)_{free-free}$.

In Figs. 5–7, the results are presented only for rigid-rigid boundaries as the results for different velocity boundary conditions change only quantitatively. Figs. 5 and 6 show the variation of $R_m$ and $a_c$ as a function of $N_S$ for different values of modified particle density increment $N_B$ (with $Da = 0.8$, $\zeta = 0.8$, $\eta = 0.6$) and the Darcy number $Da$ (with $N_B = 0.01$, $\zeta = 0.3$, $\eta = 0.8$) for $N_A = 2$, $R_m = 2$ and

Figure 4  Variation of (a) $R_m$ and (b) $a_c$ with $N_S$ for different values of $Le$ when $N_A = 2$, $R_m = 2$, $N_B = 0.01$, $Da = 0.8$, $\zeta = 0.8$, $\eta = 0.6$.

Figure 5  Variation of (a) $R_m$ and (b) $a_c$ with $N_S$ for different values of $N_B$ when $N_A = 2$, $R_m = 2$, $Le = 10$, $Da = 0.8$, $\zeta = 0.8$, $\eta = 0.6$.
Le = 10. From Fig. 5(a) and (b) it is evident that the parameter \( N_B \) has insignificant effect on the onset of convection. Fig. 6(a) reveals that increasing \( Da \) is to delay the onset of convection due to increase in the viscous diffusion. The convection cells get enlarged with increasing \( Da \) and this is evident from Fig. 6(b).

The impact of anisotropy parameters on the onset of convection is depicted in Fig. 7. The variation of \( R_{tc} \) and \( a_c \) is shown in Fig. 7(a) and (b) as a function of mechanical anisotropy parameter \( \eta \) for different values of \( N_S \) and thermal anisotropy parameter \( \xi \). The effect of decreasing \( \xi \) and \( \eta \) is to respectively slow down and accelerate the onset of convection. This may be attributed to the fact that decrease in \( \xi \) corresponds to smaller horizontal permeability which in turn hinders the motion of the fluid in the horizontal direction. As a consequence, the conduction process in the porous medium becomes more stable and hence higher values of \( R_{tc} \) are needed for the onset of convection with decreasing \( \xi \). The larger resistance to horizontal flow also leads to a shortening of the horizontal wavelength (i.e., increase in the horizontal wave number with decreasing \( \xi \)) at the onset of convection. The results presented in Fig. 7(b) corroborate this fact. In the same figures, it is also seen that for a fixed value of \( \xi \), \( R_{tc} \) decreases with decreasing \( \eta \). This is due to the fact that, as \( \eta \) decreases the horizontal thermal diffusivity also decreases. Thus heat cannot be transported through the porous layer and hence the horizontal temperature variations in the fluid required to sustain convection are less efficiently dissipated for small \( \eta \). Hence, the base state becomes less stable leading to lower values of \( R_{tc} \) and also the onset of convection occurs at a higher
wave number. The results obtained for the case of Darcy porous medium are displayed in Fig. 8. The variation of $R_{tc}$ and $a_c$ as a function of $n$ is shown in Fig. 8(a) and (b), respectively for different values of $N_S$ and for two values of $g = 0.1$ and $0.5$ when $N_A = 2$, $R_n = 2$, $Le = 1$ and $N_B = 0.01$. The critical stability parameters monotonically decrease with increasing $n$ and increase with increasing $g$. Except for a quantitative change, the qualitative behavior of the results remains the same for the Darcy and Brinkman porous media cases.

Fig. 9(a) and (b) respectively displays the variation of $R_{tc}$ with $Da$ for (a) regular fluid ($N_A = 0$, $N_B = 0$) and (b) nanofluid with $N_A = 2$ and $N_B = 0.01$ for two values of $N_S$ when $Le = 2$, $R_n = 1$ for rigid-rigid boundaries. Such that the ratio $\xi/\eta = 1$. The results for $Da = 0$ correspond to the case of Darcy porous medium. From Fig. 9(a), it is observed that the onset of convection is independent of values of anisotropic parameters $\xi$ and $\eta$ with $\xi/\eta = 1$ in the case of regular fluid saturating a Darcy porous medium, but with increasing $Da$ it is seen that the onset of convection is delayed for $\xi = 1 = \eta$ (isotropic case) compared to $\xi = 0.5 = \eta$ (anisotropic case). In contrast to this, the onset of convection is influenced at $Da = 0$ for nanofluids and the same is evident from Fig. 9(b). From the figure it is obvious that the onset of convection depends on the values of $\xi$ and $\eta$ even when they assume the same value. The system is more stabilizing for $\xi = 0.5 = \eta$ compared to $\xi = 1 = \eta$ up to certain values of $Da$ and exceeding which the trend gets reversed.
6. Conclusions

The onset of penetrative Brinkman convection via uniform internal heating is investigated in a horizontal layer of an anisotropic porous medium saturated by a nanofluid. The problem has been analyzed for three different types of boundary combinations, namely rigid-rigid, free-free and lower-rigid and upper-free which are isothermal and the flux of volume fraction of nanoparticles is zero on the boundaries. The resulting eigenvalue problem is solved numerically using the Galerkin technique.

The results of the forgoing study may be summarized as follows:

(i) The principle of exchange of stability is found to be true for all types of velocity boundary conditions. There is a qualitative agreement among the results of different velocity boundary conditions. The system is more stable when both boundaries are rigid, while the free boundaries are least stable.

(ii) The effect of increase in dimensionless heat source strength $NS$, mechanical anisotropy parameter $n$, modified diffusivity ratio $NA$, nanoparticle concentration $Darcy-Rayleigh number $Rn$ and Lewis number $Le$ is to hasten the onset of convection. There is no observable effect of modified particle density increment $NB$ on the onset of convection.

(iii) Increase in the Darcy number $Da$ and thermal anisotropy parameter $\eta$ is to delay the onset of convection.

(iv) In contrast to the regular fluid saturating a Darcy porous medium, the onset of convection is influenced for values of $N_x$ with Char and Chiang [38] for regular fluid in the absence of porous media.

| $NS$ | Rigid-rigid | | | | | Rigid-free | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Char and Chiang [38] | Present study | | | | | Char and Chiang [38] | Present study | | | | |
| $R_c$ | $a_c$ | $R_c$ | $a_c$ | $R_c$ | $a_c$ | $R_c$ | $a_c$ |
| 0 | 1707.853 | 3.116 | 1707.768 | 3.116 | 1100.684 | 2.682 | 1100.6496 | 2.682 |
| 0.5 | 1704.619 | 3.119 | 1704.5265 | 3.119 | 1055.612 | 2.679 | 1055.5792 | 2.679 |
| 1 | 1695.048 | 3.127 | 1694.9502 | 3.127 | 1011.471 | 2.680 | 1011.4394 | 2.680 |
| 5 | 1463.053 | 3.304 | 1462.9609 | 3.304 | 725.639 | 2.773 | 725.6020 | 2.733 |
| 10 | 1118.666 | 3.529 | 1118.4301 | 3.529 | 517.870 | 2.803 | 517.8309 | 2.803 |
| 15 | 878.525 | 3.659 | 878.3034 | 3.659 | 398.695 | 2.849 | 398.6599 | 2.849 |
| 30 | 521.558 | 3.819 | 521.4032 | 3.819 | 233.637 | 2.916 | 233.6116 | 2.916 |
| 40 | 408.684 | 3.863 | 408.5577 | 3.863 | 182.746 | 2.937 | 182.7257 | 2.937 |
| 70 | 247.155 | 3.994 | 247.0751 | 3.994 | 110.369 | 2.967 | 110.3557 | 2.967 |
| 100 | 176.994 | 3.994 | 176.9359 | 3.994 | 79.020 | 2.980 | 79.0101 | 2.980 |

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Table 1: Comparison of critical Rayleigh number and wave number for different values of $NS$ with Char and Chiang [38] for regular fluid in the absence of porous media.

Table 2: Comparison of regular fluid critical Rayleigh number $R_c$ and $a^2_c$ for different values of $Da$ in the case of both boundaries rigid with Guo and Kaloni [39].

References

Penetrative Brinkman convection in an anisotropic porous layer


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