Research Article

$D_a$-Homothetic Deformation of $K$-Contact Manifolds

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We study $D_a$-homothetic deformations of $K$-contact manifolds. We prove that $D_a$-homothetically deformed $K$-contact manifold is a generalized Sasakian space form if it is conharmonically flat. Further, we find expressions for scalar curvature of $D_a$-homothetically deformed $K$-contact manifolds.

1. Introduction

In 1968 Tanno [1] introduced the notion of $D_a$-homothetic deformations. Carriazo and Martín-Molina [2] studied $D_a$-homothetic deformation of generalized $(k, \mu)$ space forms and gave several examples for manifolds of dimension 3. De and Ghosh [3] studied $D_a$-homothetic deformation of almost normal contact metric manifolds and proved that $Q\phi \phi Q$ is invariant under such transformation. Bagewadi and Venkatesha [4] studied concircularly semisymmetric trans-Sasakian manifolds and De et al. [5] studied conharmonically semisymmetric, con harmonically flat, $\xi$-conharmonically flat, and conharmonically recurrent generalized Sasakian space forms. Several authors [6–11] studied $K$-contact manifolds and proved conditions for these manifolds to be $\xi$-conformally flat, $\phi$-conformally flat, quasi-conharmonically flat, and $\xi$-conharmonically flat. Motivated by the above studies, in this paper we study $D_a$-homothetic deformations of $K$-contact manifolds by considering conharmonic and projective curvature tensor. The paper is organized as follows. After Preliminaries, we give a brief account of information of $D_a$-homothetic deformation of $K$-contact manifolds in Section 3. In Section 4, we study conharmonically flat, semisymmetric, $\phi$-conharmonically flat, quasi-conhar monically flat, and $\xi$-conharmonically flat $K$-contact manifolds with respect to $D_a$-homothetic deformation. In the last section, we consider Weyl projective curvature in $K$-contact manifolds with respect to $D_a$-homothetic deformation.

2. Preliminaries

Let $(M, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact metric manifold [12], consisting of a $(1, 1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$, and Riemannian metric $g$. Then

\[ \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \]

\[ \phi \xi = 0, \quad \eta \circ \phi = 0, \]

\[ g(\phi X, \phi Y) = g(X, Y) - \eta(Y) \eta(X), \]

\[ g(\phi X, \phi Y) = -g(\phi X, Y), \]

\[ g(X, \phi Y) = 0, \quad g(X, \xi) = \eta(X), \]

for all $X, Y \in TM$. If $\xi$ is a Killing vector field, then $M$ is called a $K$-contact Riemannian manifold [13]. A $K$-contact Riemannian manifold is called Sasakian [12], if the relation

\[ (\nabla_X \phi) Y = g(X, Y) \xi - \eta(Y) X \]

holds, where $\nabla$ denotes the operator of covariant differentiation with respect to $g$. 


If $M^{2n+1}$ is a $K$-contact Riemannian manifold, then besides (1), (2), (3), and (4) the following relations hold [14]:

$$\nabla_X \xi = -\phi X,$$

(5)

$$\nabla_X \eta (Y) = -g (\phi X, Y),$$

(6)

$$S (X, \xi) = g (QX, \xi) = 2n \eta (X),$$

(7)

$$\eta (R(X, Y) Z) = g (Y, Z) \eta (X) - g (X, Z) \eta (Y),$$

(8)

$$R(X, Y) \xi = \eta (Y) X - \eta (X) Y,$$

(9)

$$R(\xi, X) Y = g (X, Y) \xi - \eta (Y) X,$$

(10)

for any vector fields $X$ and $Y$, where $R$ and $S$ denote, respectively, the curvature tensor of type $(1,3)$ and the Ricci tensor of type $(0,2)$.

**Definition 1.** A contact metric manifold $M$ is said to be $\eta$-Einstein if

$$S(X, Y) = \alpha g(X, Y) + \beta \eta (X) \eta (Y),$$

where $\alpha$ and $\beta$ are smooth functions on $M$.

### 3. $D_a$-Homothetic Deformation of $K$-Contact Manifolds

Let $(M, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact metric manifold. A $D_a$-homothetic deformation is defined by

$$\bar{\phi} = \phi, \quad \bar{\xi} = \frac{1}{a} \xi, \quad \bar{\eta} = a \eta,$$

(11)

$$\bar{g} = ag + a (a - 1) \eta \otimes \eta,$$

with $a$ being a positive constant [1].

It is clear that the $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also an almost contact metric manifold. If $(M, \phi, \xi, \eta, g)$ is a $K$-contact manifold with Riemannian connection $\nabla$, the connection $\nabla$ of the $D_a$-deformed $K$-contact manifold $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ can be calculated from $\nabla$ and $\bar{g}$.

Using Koszul's formula and (5), (6), and (11), $\nabla$ of $\bar{g}$ is given by

$$\nabla_X Y = \nabla_X Y - (a - 1) \left[ \eta (Y) \phi X + \eta (X) \phi Y \right].$$

(12)

Using (12), we obtain

$$\nabla_X \phi = (\nabla_X \phi) Y + (a - 1) \eta (Y) \phi \xi.$$ 

(13)

The curvature tensor $\bar{R}$ of $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is given by

$$\bar{R}(X, Y) Z = R(X, Y) Z - (a - 1)$$

$$\times \left[ g (\phi Y, Z) \phi X + g (\phi Z, X) \phi Y + 2g (\phi Y, X) \phi Z \right.$$

$$+ \left[ g (X, Z) \xi - \eta (X) Z \right] \eta (Y)$$

$$- \left[ g (Y, Z) \xi - \eta (Y) Z \right] \eta (X)$$

$$+ a \left[ \eta (Y) X - \eta (X) Y \right] \eta (Z) \right].$$

(14)

Using (9), (10), and (14), we have

$$\bar{R}(X, Y) \xi = (2 - a) \left[ \eta (Y) X - \eta (X) Y \right],$$

$$\bar{R}(\xi, Y) Z = [g (Y, Z) \xi - \eta (Z) Y] - (a - 1) \left[ \eta (Y) X - \eta (X) \right] \eta (Z),$$

$$\bar{R}(\xi, Y) \xi = \bar{S}(Y, Z) = a^2 \left[ g (Y, Z) \eta (X) - g (X, Z) \eta (Y) \right].$$

(15)

From (14), we get

$$\bar{S}(Y, Z) = a \bar{S}(Y, Z) - a^2 \left[ g (Y, Z) \eta (X) - g (X, Z) \eta (Y) \right],$$

(16)

Again contracting (16) over $Y$, $Z$, we get

$$\bar{r} = ar - 2na (a - 1),$$

(19)

where $\bar{r}$ and $r$ are the scalar curvatures of $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ and $(M, \phi, \xi, \eta, g)$, respectively.

### 4. Conharmonic Curvature Tensor in $D_a$-Homothetically Deformed $K$-Contact Manifolds

The conharmonic tensor of a $D_a$-homothetically deformed $K$-contact manifold is defined by [15]

$$\bar{R}(X, Y) Z = R(X, Y) Z - \frac{1}{2n - 1}$$

$$\times \left[ \bar{S}(Y, Z) X - \bar{S}(X, Z) Y + \bar{g}(Y, Z) \bar{Q}X \right.$$

$$- \bar{g}(X, Z) \bar{Q}Y \right],$$

(20)

for $X, Y, Z \in TM$, where $\bar{R}, \bar{S}$, and $\bar{Q}$ are the Riemannian curvature tensor, Ricci tensor, and Ricci operator of $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$.

**Definition 2.** An almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is said to be

(1) conharmonically flat if

$$K(X, Y) Z = 0,$$

(21)
(2) conharmonically semisymmetric if
\[ R \cdot K = 0, \]  
(22)

(3) \( \phi \)-conharmonically flat if
\[ g(K(\phi X, \phi Y) \phi Z, \phi W) = 0, \]  
(23)

(4) quasi-conharmonically flat if
\[ g(K(X, Y) Z, \phi W) = 0, \]  
(24)

(5) \( \xi \)-conharmonically flat if
\[ K(X, Y) \xi = 0, \]  
(25)

for all vector fields \( X, Y, \) and \( Z \).

Assume that \( \overline{M} \) is conharmonically flat \( K \)-contact manifold with respect to \( D_a \)-homothetic deformation. So, we have \( \overline{K}(X, Y) Z = 0 \).

Then from (20), we have
\[ \overline{R}(X, Y) Z = \frac{1}{2n-1} \times \left[ S(Y, Z) X - S(X, Z) Y + \overline{g}(Y, Z) \overline{Q} X \right. \]
\[ \left. - \overline{g}(X, Z) \overline{Q} Y \right]. \]  
(26)

Setting \( Z = \bar{\xi} \), contracting (26) with \( W \), and using (7), (9), (14), and (16), we obtain
\[ (2 - a)(2n - 1 - 2na) \left[ \eta(Y) g(X, W) - \eta(X) g(Y, W) \right] \]
\[ = \eta(Y) S(X, W) - \eta(X) S(Y, W). \]  
(27)

Taking \( Y = \bar{\xi} \) in (27) and using (1), (7), and (16), it follows that
\[ S(X, W) = \frac{(2 - a)(2n - 1 - 2na)}{a} \overline{g}(X, W) \]
\[ + \frac{(2 - a)(4na - 2n + 1)}{a} \eta(X) \eta(W). \]  
(28)

Thus, \( \overline{M} \) is \( \eta \)-Einstein.

Using (28) in (26), we obtain
\[ \overline{R}(X, Y, Z, W) \]
\[ = \frac{2(2 - a)(2n - 1 - 2na)}{a(2n - 1)} \times \left[ \overline{g}(Y, Z) \overline{g}(X, W) - \overline{g}(X, Z) \overline{g}(Y, W) \right] \]
\[ + \frac{(2 - a)(4na - 2n + 1)}{a(2n - 1)} \times \left[ \left( \overline{g}(X, W) \overline{g}(Y) - \overline{g}(Y, W) \overline{g}(X) \right) \overline{g}(Z) \right. \]
\[ \left. + \left[ \overline{g}(Y, Z) \overline{g}(X) - \overline{g}(X, Z) \overline{g}(Y) \right] \eta(W) \right]. \]  
(29)

From (29), we get
\[ \overline{R}(X, Y) Z = \frac{2(2 - a)(2n - 1 - 2na)}{a(2n - 1)} \times \left[ \overline{g}(Y, Z) X - \overline{g}(X, Z) Y \right] \]
\[ - \frac{(2 - a)(4na - 2n + 1)}{a(2n - 1)} \times \left[ \overline{g}(X) \overline{g}(Z) Y - \overline{g}(Y) \overline{g}(Z) X \right. \]
\[ \left. + \overline{g}(X, Z) \overline{g}(Y) \eta(Z) - \overline{g}(Y, Z) \overline{g}(X) \eta(X) \right]. \]  
(30)

Hence, it reduces to a generalized Sasakian space form with \( f_1 = 2(2 - a)(2n - 1 - 2na)/a(2n - 1), f_2 = 0, \) and \( f_3 = -(2 - a)(4na - 2n + 1)/a(2n - 1). \) Thus, (30) leads to the following.

**Theorem 3.** A conharmonically flat \( K \)-contact manifold admitting \( D_a \)-homothetic deformation reduces to a generalized Sasakian space form with associated functions \( f_1 = 2(2 - a)(2n - 1 - 2na)/a(2n - 1), f_2 = 0, \) and \( f_3 = -(2 - a)(4na - 2n + 1)/a(2n - 1). \)

Let us now consider a conharmonically semisymmetric \( K \)-contact manifold admitting \( D_a \)-homothetic deformation. Then the condition
\[ \overline{R}(X, Y) K(U, V) Z - K(\overline{R}(X, Y) U, V) Z \]
\[ - K(U, \overline{R}(X, Y) V) Z = 0. \]  
(31)
Therefore,
\[
\mathcal{g}(\overline{R}(\xi, Y) \overline{K}(U, V) Z, \xi) - \mathcal{g}(\overline{K}(\overline{R}(\xi, Y) U, V) Z, \xi) - \mathcal{g}(\overline{K}(U, V) \overline{R}(\xi, Y) Z, \xi) = 0.
\]
(36)

From this it follows that
\[
- \overline{K}(U, V, Z, Y) + \eta \left( \overline{K}(U, V) Z \right) \eta(Y) + [g(Y, U) - (a-1) \eta(U) \eta(Y)] \eta(\overline{K}(U, \xi) Z) + (a-2) \eta(U) \eta(\overline{K}(U, Y) Z) + [g(Y, Z) - (a-1) \eta(Z) \eta(Y)] \eta(\overline{K}(U, V) \xi) + (a-2) \eta(Z) \eta(\overline{K}(U, V) Y) = 0,
\]
(37)
where
\[
\overline{K}(U, V, Z, Y) = \mathcal{g}(\overline{K}(U, V) Z, Y).
\]
(38)

Taking \( Y = U \) in (37) and making use of (32) and (33), we obtain
\[
- \overline{K}(U, V, Z, U) + (a-1) \eta(U) \eta(\overline{K}(U, V) Z) + [g(U, U) - (a-1) \eta(U) \eta(U)] \eta(\overline{K}(U, \xi) Z) + [g(U, V) - (a-1) \eta(U) \eta(V)] \eta(\overline{K}(U, V) \xi) + (a-2) \eta(Z) \eta(\overline{K}(U, V) Y) = 0.
\]
(39)

If \( \{e_1, e_2, \ldots, e_{2n}, \xi \} \) is a local orthonormal basis of vector fields in \( M \), then, from (39), we get
\[
\sum_{i=1}^{2n} \overline{K}(e_i, V, Z, e_i) = (a-1) \sum_{i=1}^{2n} \eta(e_i) \eta(\overline{K}(e_i, V) Z)
+ \sum_{i=1}^{2n} [g(e_i, e_i) - (a-1) \eta(e_i) \eta(e_i)] \eta(\overline{K}(\xi, V) Z)
+ \sum_{i=1}^{2n} [g(e_i, V) - (a-1) \eta(e_i) \eta(V)] \eta(\overline{K}(e_i, \xi) Z)
+ (a-2) \sum_{i=1}^{2n} \eta(\overline{K}(e_i, V) e_i) \eta(Z).
\]

From (20), it follows that
\[
\sum_{i=1}^{2n} \overline{K}(e_i, V, Z, e_i) = \frac{1}{2n-1} \overline{S}(V, Z)
- \left[ 1 - \frac{\overline{r} + 2n(2-a)}{2n-1} \right] \mathcal{g}(V, Z)
+ \left[ 1 - \frac{4n(2-a)}{2n-1} \right] \overline{\eta}(V) \overline{\eta}(Z),
\]
(40)

Using (41) in (40), we obtain
\[
(2n + 2 - a) \eta(\overline{K}(\xi, V) Z)
= \frac{(a-1) \left( a^2 - 1 \right)}{a^2 (2n-1)} \overline{S}(V, Z)
+ \left[ \frac{\overline{r} - 2na(a+1)(a^2 - a + 1)}{a^2 (2n-1)} \right] \mathcal{g}(V, Z)
+ \left[ \frac{(a-2) \overline{r}}{a^2 (2n-1)} + \frac{p}{a^2 (2n-1)} \right] \overline{\eta}(V) \overline{\eta}(Z),
\]
(42)
where
\[
p = -2na^5 + 2na^4 + 10na^3 + a^2 (8n^2 - 18n)
+ a (14n - 24n^2 + 1) + 16n^2 - 6n - 1.
\]
(43)
In view of (34), (42) yields
\[ S(V, Z) = - \frac{1}{a^2 (2n + 1) + 1 - a} \left[ a \mathcal{G}(V, Z) + \beta \mathcal{H}(V) \mathcal{H}(Z) \right], \tag{44} \]
where
\[ \alpha = r a^2 + 2 n a^4 + a^3 \left(4n^2 - 6n - 1\right) \]
\[ + a^2 \left(4n^5 + 8n + 2\right) + a \left(-1 - 4n\right) + 1 + 2n, \]
\[ \beta = \left(a - 2\right) r - 2 n a^4 + a^3 \left(8n^2 + 24n + 1\right) \tag{45} \]
\[ + a^2 \left(-32n^2 - 32n - 2\right) + a \left(14n - 6n^2\right) + 1 + 2n^2 - 6n - 1. \]

Thus, \( M \) is \( \eta \)-Einstein.

If \( \{e_1, e_2, \ldots, e_{2n}, \xi\} \) is a local orthonormal basis of vector fields in \( M \), then, from (44), we get
\[ \bar{r} = \left(-2 n a^5 + 4 n^2 a^4 + a^3 l + a^2 m \right) \]
\[ + a \left(-32 n^3 + 8n + 20n^2 + 2n\right) \]
\[ \times \left(1 - 2a^2 (2n + 1) \right)^{-1}, \tag{46} \]
where
\[ l = -8 n^3 - 8n^2 + 16n, \]
\[ m = 8 n^3 + 16n^2 - 20n. \tag{47} \]

So, we can state the following.

**Theorem 4.** In a \((2n + 1)\)-dimensional conharmonically semisymmetric \( K \)-contact manifold admitting \( D_a \)-homothetic deformation, scalar curvature \( \bar{r} \) is given by (46).

Analogous to the definition of \( \phi \)-conharmonically flat \( K \)-contact manifolds [8], we define \( \phi \)-conharmonically flat \( K \)-contact manifolds with respect to \( D_a \)-homothetic deformation. Let us assume that \( M \) is a \( \phi \)-conharmonically flat \( K \)-contact manifold with respect to \( D_a \)-homothetic deformation. It can be easily seen that
\[ \mathcal{G}(\bar{K}(\phi X, \phi Y) \phi Z, \phi W) = 0, \tag{48} \]
where \( X, Y, Z, W \in TM \).

Using (20), (48) yields
\[ \mathcal{G}(\bar{R}(\phi X, \phi Y) \phi Z, \phi W) = \frac{1}{2n - 1} \left( \mathcal{S}(\phi Y, \phi Z) \mathcal{G}(\phi X, \phi W) \right. \]
\[ - \mathcal{S}(\phi X, \phi Z) \mathcal{G}(\phi Y, \phi W) \]
\[ + \mathcal{S}(\phi X, \phi W) \mathcal{G}(\phi Y, \phi Z) \]
\[ - \mathcal{S}(\phi Y, \phi W) \mathcal{G}(\phi X, \phi Z) \bigg), \tag{49} \]
for all \( X, Y, Z, W \in TM \).

If \( \{e_1, e_2, \ldots, e_{2n}, \xi\} \) is a local orthonormal basis of vector fields in \( \overline{M} \), then \( \{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\} \) is also a local orthonormal basis. So, using (1), (6), (14), (16), and (18), it can be easily verified that
\[ \sum_{i=1}^{2n} \mathcal{R}(\phi e_i, \phi Y, \phi Z, \phi e_i) = \mathcal{S}(Y, Z) - a g(Y, Z) \]
\[ + \left[a - 2 n a \left(2 - a\right)\right] \eta(Y) \eta(Z), \]
\[ \sum_{i=1}^{2n} \mathcal{G}(\phi e_i, \phi e_i) = 2n, \]
\[ \sum_{i=1}^{2n} \mathcal{S}(\phi e_i, \phi e_i) = \bar{r} - 2 n \left(2 - a\right), \]
\[ \sum_{i=1}^{2n} \mathcal{S}(\phi Y, \phi e_i) \mathcal{G}(\phi e_i, \phi Z) = \mathcal{S}(\phi Y, \phi Z). \tag{50} \]

For a local orthonormal basis \( \{\phi e_1, \phi e_2, \ldots, \phi e_{2n}, \xi\} \) of vector fields in \( \overline{M} \), putting \( X = W = e_i \) in (49) and summing up with respect to \( i = 1, 2, \ldots, 2n + 1 \), we have
\[ \frac{1}{2n - 1} \sum_{i=1}^{2n} (\mathcal{S}(\phi Y, \phi Z) \mathcal{G}(\phi e_i, \phi e_i) \]
\[ - \mathcal{S}(\phi X, \phi Z) \mathcal{G}(\phi Y, \phi e_i) \]
\[ + \mathcal{S}(\phi X, \phi e_i) \mathcal{G}(\phi Y, \phi Z) \]
\[ - \mathcal{S}(\phi Y, \phi e_i) \mathcal{G}(\phi X, \phi Z) \bigg) \tag{51} \]
for all \( Y, Z \in T \overline{M} \).

The previous equation, in view of (50), becomes
\[ \mathcal{S}(\phi Y, \phi Z) = \left[a \left(2n - 1\right) + a \left(\bar{r} - 2 n \left(2 - a\right)\right)\right] \]
\[ \times \left[ g(Y, Z) - \eta(Y) \eta(Z) \right], \tag{52} \]
for all \( Y, Z \in T \overline{M} \).

Using (2) and (18), (52) reduces to
\[ \mathcal{S}(Y, Z) = \left[\bar{r} + 2 n - 1 - 2 n \left(2 - a\right)\right] g(Y, Z) \]
\[ + \left[2 n \left(2 - a\right)\left(a^2 + 2 a - 1\right) - (2a - 1) (\bar{r} + 2 n - 1)\right] \eta(Y) \eta(Z). \tag{53} \]

Setting \( Y = Z = e_i \) in (53), summing up with respect to \( i = 1, 2, \ldots, 2n + 1 \), and using (19), we obtain
\[ \bar{r} = \frac{-2 a^3 + a \left(4n^2 + 8n + 2\right) - 4n^2 - 6n - 2}{2a - 2n - 1}. \tag{54} \]
Replacing $X$ by $\phi X$ and $Y$ by $\phi Y$ in (53) and using (54), we obtain $S(\phi X, \phi Y) = (r + 2n - 1 - 2n(2 - a))\mathcal{G}(\phi X, \phi Y)$ for all $X, Y \in T\mathcal{M}$.

Now using the previous expression in (49), we obtain

$$\mathcal{R}(\phi X, \phi Y, \phi Z, \phi W) = 2\left(\frac{r + 2n - 1 - 2n(2 - a)}{2n - 1}\right) \times (\mathcal{G}(\phi Y, \phi Z) \mathcal{G}(\phi X, \phi W) - \mathcal{G}(\phi X, \phi Z) \mathcal{G}(\phi Y, \phi W)), \quad (55)$$

for all $X, Y, Z, W \in T\mathcal{M}$.

The converse is obvious. Thus we have the following.

**Theorem 5.** A $(2n + 1)$-dimensional $K$-contact manifold is $\phi$-conharmonically flat with respect to $D_\alpha$-homothetic deformation if and only if $\mathcal{M}$ satisfies (55).

From (20), we obtain

$$\mathcal{G}(\mathcal{K}(X, Y) Z, \phi W) = \mathcal{R}(X, Y, Z, \phi W) - \frac{1}{2n - 1} \times (\mathcal{S}(Y, Z) \mathcal{G}(X, \phi W) - \mathcal{S}(X, Z) \mathcal{G}(Y, \phi W)) + \mathcal{S}(X, \phi W) \mathcal{G}(Y, Z) - \mathcal{S}(Y, \phi W) \mathcal{G}(X, Z)), \quad (56)$$

for all $X, Y, Z, W \in T\mathcal{M}$.

Suppose that $\mathcal{M}$ is quasi-conharmonically flat $K$-contact manifold with respect to $D_\alpha$-homothetic deformation; that is,

$$\mathcal{G}(\mathcal{K}(X, Y) Z, \phi W) = 0. \quad (57)$$

Then (56) reduces to

$$\mathcal{R}(X, Y, Z, \phi W) = \frac{1}{2n - 1} \left(\mathcal{S}(Y, Z) \mathcal{G}(X, \phi W) - \mathcal{S}(X, Z) \mathcal{G}(Y, \phi W)\right) + \mathcal{S}(X, \phi W) \mathcal{G}(Y, Z) - \mathcal{S}(Y, \phi W) \mathcal{G}(X, Z)). \quad (58)$$

For a local orthonormal basis $\{e_1, e_2, \ldots, e_{2n}, \xi\}$ of vector fields in $\mathcal{M}$, putting $X = \phi e_i$ and $W = e_i$ in (58) and summing up with respect to $i = 1, 2, \ldots, 2n + 1$, we obtain

$$\sum_{i=1}^{2n} \mathcal{R}(\phi e_i, Y, Z, e_i) = \frac{1}{2n - 1} \left(\mathcal{S}(Y, Z) \mathcal{G}(\phi e_i, \phi e_i) - \mathcal{S}(\phi e_i, Z) \mathcal{G}(Y, \phi e_i)\right) + \mathcal{S}(\phi e_i, \phi e_i) \mathcal{G}(Y, Z) - \mathcal{S}(Y, \phi e_i) \mathcal{G}(\phi e_i, Z)). \quad (59)$$

Using (2), (10), (14), and (17) in (59), we obtain

$$\mathcal{S}(Y, Z) = \left[\frac{r + 2n - 1 - 2n(2 - a)}{2n - 1}\right] \frac{\mathcal{G}(Y, Z)}{\mathcal{G}(\phi e_i, \phi e_i)}.$$  

Taking $Z = \xi$ and using (1) and (17), we obtain

$$\mathcal{S}(Y, Z) = \frac{2n(2 - a)(a + 1)}{a}, \quad (61)$$

and using (61) in (19), we obtain

$$r = \frac{4n}{a}. \quad (62)$$

Hence, we can state the following.

**Theorem 6.** Let $M$ be a $K$-contact manifold. Suppose that $\mathcal{M}$ is obtained from $M$ by $D_\alpha$-homothetic deformation. If $\mathcal{M}$ is quasi-conharmonically flat, then the scalar curvatures $\mathcal{S}(Y, Z)$ of $M$ and $M$ are, respectively, given by (61) and (62).

Suppose that $\mathcal{M}$ is $\xi$-conharmonically flat. Then from (20), we have

$$\mathcal{R}(X, Y, \xi, W) = \frac{1}{2n - 1} \left(\frac{2n(2 - a)}{a} \left[\eta(Y) X - \eta(X) Y\right]\right) \mathcal{G}(X, W) - \mathcal{S}(X, W) \mathcal{G}(\xi, \phi W). \quad (63)$$

Contracting the above equation with respect to $W$, we obtain

$$\mathcal{R}(X, Y, \xi, W) = \frac{1}{2n - 1} \left(\frac{2n(2 - a)}{a} \left[\eta(Y) \mathcal{G}(X, W) - \eta(X) \mathcal{G}(Y, W)\right]\right) + \mathcal{S}(X, W) \mathcal{G}(\xi, \phi W)). \quad (64)$$

for all $X, Y, Z, W \in T\mathcal{M}$.

For a local orthonormal basis $\{e_1, e_2, \ldots, e_{2n}, \xi\}$ of vector fields in $\mathcal{M}$, using (64), we obtain

$$\sum_{i=1}^{2n} \mathcal{R}(e_i, Y, \xi, e_i) = \frac{1}{2n - 1} \left(\frac{2n(2 - a)}{a} \left[\eta(Y) \mathcal{G}(e_i, e_i) - \eta(e_i) \mathcal{G}(Y, e_i)\right]\right) + \mathcal{S}(e_i, e_i) \mathcal{G}(\xi, \phi W)). \quad (65)$$

Therefore,

$$2n\mathcal{S}(Y, \xi) = \frac{4n^2(2 - a) + \mathcal{R}}{\eta(Y)}. \quad (66)$$

Using (17) in (66), we obtain

$$\mathcal{R} = 0. \quad (67)$$
Taking $Y = \xi$ in (64) and using (10), (14), and (17), we obtain
\[
\bar{S}(X, W) = [2na - 4n - 1] \bar{g}(X, W) + [6n - 4n + 1] \bar{\eta}(X) \bar{\eta}(W).
\]
(68)
From this we can conclude that $M$ is $\eta$-Einstein. Thus, we have the following.

**Theorem 7.** A $D_a$-Homothetically deformed $\xi$-conharmonically flat $K$-contact manifold is $\eta$-Einstein and its scalar curvature vanishes.

5. **$D_a$-Homothetic Deformation of $\xi$-Weyl Projectively Flat $K$-Contact Manifolds**

Suppose that, in a $(2n + 1)$-dimensional $K$-contact manifold $M$ with $D_a$-homothetic deformation, the Ricci tensor vanishes; that is,
\[
\bar{S}(X, Y) = 0.
\]
(69)
Then from (16), we have
\[
S(Y, Z) = (a - 1) [3g(\phi Y, \phi Z) - (1 + a) g(Y, Z)
+ [2n(a - 1) + 1 + a] \eta(Y) \eta(Z)].
\]
(70)
The Weyl projective curvature tensor of $M$ is given by [16]
\[
W(X, Y, Z) = R(X, Y) Z - \frac{1}{2n} [S(Y, Z) X - S(X, Z) Y].
\]
(71)
If $W(X, Y) \xi = 0$, then (71) reduces to
\[
R(X, Y) \xi = \frac{1}{2n} [S(Y, \xi) X - S(X, \xi) Y].
\]
(72)
Using (70) in (72), we obtain
\[
R(X, Y) \xi = (a - 1)^2 [\eta(Y) X - \eta(X) Y].
\]
(73)
The Weyl projective curvature tensor of $\bar{M}$ with respect to $D_a$-homothetic deformation is given by
\[
\bar{W}(X, Y, Z) = R(X, Y) Z - \frac{1}{2n} [\bar{S}(Y, Z) X - \bar{S}(X, Z) Y].
\]
(74)
Now using (14) and (69) in (74), we get
\[
\bar{W}(X, Y) \xi = R(X, Y) \xi - \frac{(a - 1)^2}{a} [\eta(Y) X - \eta(X) Y].
\]
(75)
From (73), the (75) reduces to
\[
\bar{W}(X, Y) \xi = 0.
\]
(76)
Thus, we can state the following.

**Theorem 8.** Let $\bar{M}$ be obtained from a $K$-contact manifold $M$ by $D_a$-homothetic deformation. If the Ricci tensor of $\bar{M}$ vanishes, then it is $\xi$-Weyl projectively flat.