A Note on Cubic Modular Equations of Degree Two*

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Abstract

On Page 259 of his second notebook [3], Ramanujan recorded many cubic modular equations of degree 2. In this paper we establish several cubic modular equations of degree 2 akin to those in Ramanujan’s work. As an application of our results, we also establish some new $P - Q$ eta-function identities.

Keywords and Phrases: Cubic modular equations, Eta-function identities.

1. A Family of Cubic Modular Equations

The ordinary hypergeometric series $\,_{2}F_{1}(a,b;c;x)$ is defined by

$$\,_{2}F_{1}(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n},$$

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(a)_0 = 1, (a)_n = a(a+1)(a+2)...(a+n-1), for n ≥ 1, |x| < 1.

Let

\[ Z(r) := Z(r; x) := _2 F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; x \right) \]

and

\[ q_r := q_r(x) := \exp \left( -\pi \csc \left( \frac{\pi}{r} \right) \frac{_2 F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; 1-x \right)}{_2 F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; x \right)} \right) , \]

where \( r = 2, 3, 4, 6 \) and \( 0 < x < 1 \).

Let \( n \) denote a fixed natural number, and assume that

\[ n \frac{_2 F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha \right)}{_2 F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; \alpha \right)} = \frac{_2 F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta \right)}{_2 F_1 \left( \frac{1}{r}, \frac{r-1}{r}; 1; \beta \right)} , \tag{1.1} \]

where \( r = 2, 3, 4 \) or 6. Then a modular equation of degree \( n \) in the theory of elliptic functions of signature \( r \) is a relation between \( \alpha \) and \( \beta \) induced by (1.1).

On Pages 257-262 of his second notebook [3, pp. 257-262], Ramanujan gives an outline of the theories of elliptic functions to alternate bases corresponding to the classical theory by way of statements of some results. Venkatachaliengar [4] examined some of these results. Proofs of all these identities can be found in [2, pp.122-123]. Recently, Adiga, Kim and Naika [1] also established some cubic modular equations in the theory of signature 3. Now we state a transformation formula which is useful in establishing several cubic equations of degree 2 in the theory of signature 3.
Lemma 1.1. (see [3, p. 258]). If
\[
\alpha := \alpha(q) = \frac{p(3 + p)^2}{2(1 + p)^3} \text{ and } \beta := \beta(q) = \frac{p^2(3 + p)}{4},
\]
(1.2)
then for \(0 \leq p \leq 1\),
\[
_{2}F_{1}\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right) = (1 + p)_{2}F_{1}\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right).
\]
(1.3)
For a proof of Lemma 1.1, see the work of Berndt [2, p. 112].

Theorem 1.1. If \(\beta\) is of degree 2 over \(\alpha\) in the theory of signature 3, then

(i)
\[
m^3 = 3 \left(\frac{\beta(1 - \beta)}{\alpha(1 - \alpha)}\right)^{\frac{1}{3}} \left(\left(\frac{1 - \beta}{\alpha}\right)^{\frac{1}{3}} - \left(\frac{\beta}{1 - \alpha}\right)^{\frac{1}{3}}\right) + \frac{8}{m^3} \left(\frac{\beta(1 - \beta)}{\alpha(1 - \alpha)}\right),
\]
(1.4)

(ii)
\[
m^2 \left(\frac{\alpha(1 - \alpha)}{\beta^2(1 - \beta)^2}\right)^{\frac{1}{3}} = m^6 \left(\frac{\alpha(1 - \alpha)}{\beta(1 - \beta)}\right) + \frac{4}{3},
\]
(1.5)

(iii)
\[
m^4 \left(\frac{\beta(1 - \beta)}{\alpha^2(1 - \alpha)^2}\right)^{\frac{1}{3}} = 16 \left(\frac{\beta(1 - \beta)}{\alpha(1 - \alpha)}\right) + \frac{m^6}{3},
\]
(1.6)

(iv)
\[
\frac{8}{m^3} = \frac{\alpha}{\beta} - 3 \left(\frac{\alpha(1 - \alpha)^2}{\beta^2(1 - \beta)}\right)^{\frac{1}{3}},
\]
(1.7)

(v)
\[
m^3 = \frac{1 - \beta}{1 - \alpha} - 3 \left(\frac{\beta^2(1 - \beta)}{\alpha(1 - \alpha)^2}\right)^{\frac{1}{3}},
\]
(1.8)
(vi) 
\[ m^3 = 3 \left( \frac{\beta(1 - \beta)^2}{\alpha^2(1 - \alpha)} \right)^{\frac{1}{3}} - \frac{\beta}{\alpha}, \quad (1.9) \]

(vii) 
\[ \frac{8}{m^3} = 3 \left( \frac{\alpha^2(1 - \alpha)}{\beta(1 - \beta)^2} \right)^{\frac{1}{3}} - \frac{1 - \alpha}{\beta}, \quad (1.10) \]

(viii) 
\[ m = 3 \left( \frac{\beta}{\alpha^2} \right)^{\frac{1}{3}} - \frac{4 \beta}{m^2 \alpha}, \quad (1.11) \]

(ix) 
\[ m^2 = 3 \left( \frac{1 - \alpha}{(1 - \beta)^2} \right) - \frac{2}{m} \left( \frac{1 - \beta}{1 - \alpha} \right), \quad (1.12) \]

(x) 
\[ \left( \frac{\beta(1 - \alpha)^2}{\alpha^2(1 - \beta)} \right)^{\frac{1}{3}} = \left( \frac{(\alpha(1 - \beta)^2)^{\frac{1}{3}} - 3(\beta^2(1 - \alpha))^{\frac{1}{3}}}{3(\alpha(1 - \beta)^2)^{\frac{1}{3}} - (\beta^2(1 - \alpha))^{\frac{1}{3}}} \right) \quad (1.13) \]

and

(xi) 
\[ \left( \frac{\alpha(1 - \beta)^2}{\beta^2(1 - \alpha)} \right)^{\frac{1}{3}} = \left( \frac{(\beta(1 - \alpha)^2)^{\frac{1}{3}} - 3(\alpha^2(1 - \beta))^{\frac{1}{3}}}{3(\beta(1 - \alpha)^2)^{\frac{1}{3}} - (\alpha^2(1 - \beta))^{\frac{1}{3}}} \right). \quad (1.14) \]

**Proof of (1.4).** From (1.2), by elementary calculations, we have

\[ 1 - \alpha = \frac{(1 - p)^2(1 + p)}{2(1 + p)^3} \quad \text{and} \quad 1 - \beta = \frac{(1 - p)(2 + p)^2}{4} \quad (1.15) \]

Using (1.2) and (1.15) in (1.4), we find that

\[ 3 \left( \frac{\beta(1 - \beta)}{\alpha(1 - \alpha)} \right)^{\frac{1}{3}} \left( \left( \frac{1 - \beta}{\alpha} \right)^{\frac{1}{3}} - \left( \frac{\beta}{1 - \alpha} \right)^{\frac{1}{3}} \right) + \frac{8}{m^3} \left( \frac{\beta(1 - \beta)}{\alpha(1 - \alpha)} \right) \]
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\[(1 + p)^3 = m^3.\]  

This completes the proof of (1.4).

The proofs of the identities (1.5) to (1.15) are similar to the proof of (1.4). We omit the details.

2. P-Q Eta-Function Identities

Following Ramanujan’s work, we define

\[\varphi(q) = f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2},\]

\[\psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\]

and

\[f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2-1}{2}}\]

where

\[(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n), \ |q|<1.\]

In this section we obtain some new P−Q eta-function identities on employing modular equations in Section 2 and the following lemma:

**Lemma 2.1.** For \(0 < x < 1,\)

\[b(q) = (1 - x)^{\frac{1}{3}} z = \frac{f^3(-q)}{f(-q^3)} \text{ and } c(q) = x^{\frac{1}{3}} z = \frac{3q^{\frac{1}{6}} f^3(-q^3)}{f(-q)}. \quad (2.1)\]
For a proof of Lemma 2.1, see [2, p.109].

**Theorem 2.1.** (see [3, p. 327]). Let

\[
P = \frac{f(-q^2)}{q^{\frac{1}{2}} f(-q^3)} \quad \text{and} \quad Q = \frac{f(-q)}{q^{\frac{3}{2}}}. \tag{2.2}
\]

Then

\[
(PQ)^2 - 9 \frac{(PQ)^2}{(PQ)^2} = \left( \frac{Q}{P} \right)^3 - \left( \frac{P}{Q} \right)^3. \tag{2.3}
\]

**Proof.** Using (2.1) in (1.4) and then using (2.2), we obtain

\[
1 = \frac{P^5}{Q} + \frac{9P}{Q^5} + \frac{8P^6}{Q^6}. \tag{2.4}
\]

On simplification, we obtain (2.3).

**Theorem 2.2.** Let

\[
P = \frac{\psi^4(q)}{q \psi^4(q^3)} \quad \text{and} \quad Q = \frac{\psi^4(q^2)}{q^2 \psi^4(q^6)} . \tag{2.5}
\]

Then

\[
P^2 \left( \frac{P - 9}{P - 1} \right) = Q \left( \frac{Q - 9}{Q - 1} \right)^2. \tag{2.6}
\]

**Proof.** Using (2.1) in (1.13), we find that

\[
\frac{\varphi^4(-q)}{\varphi^4(-q^3)} = \frac{\psi^4(q) - 9q \psi^4(q^3)}{\psi^4(q) - q \psi^4(q^3)}. \tag{2.7}
\]
Using Entry 24(ii) and (iv) of Chapter 16 of Ramanujan’s second notebook [3, p. 198] in (2.7), we obtain

\[
\frac{f^6(-q)}{q^2 f^6(-q^3)} = \frac{\psi^2(q)}{q^2 \psi^2(q^3)} \frac{\psi^4(q) - 9q\psi^4(q^3)}{\psi^4(q) - q\psi^4(q^3)}
\]  
(2.8)

and

\[
\frac{f^{12}(-q^2)}{f^{12}(-q^6)} = \frac{\psi^8(q)}{\psi^8(q^3)} \left( \frac{\psi^4(q) - 9q\psi^4(q^3)}{\psi^4(q) - q\psi^4(q^3)} \right)
\]  
(2.9)

Using (2.5) in (2.8) and (2.9), we obtain the required result.

**Theorem 2.3.** Let

\[
P = \frac{\varphi(-q)}{\varphi(-q^3)} \quad \text{and} \quad Q = \frac{\varphi(-q^2)}{\varphi(-q^6)}.
\]  
(2.10)

Then

\[
P \left( \frac{P - 9}{P - 1} \right)^2 = Q^2 \left( \frac{Q - 9}{Q - 1} \right).
\]  
(2.11)

The proof of Theorem 2.3 is similar to the proof of Theorem 2.2, so we omit the details.

**Remark.** The $P - Q$ eta-function identities (2.6) and (2.12) appear to be new in the literature.

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References


