On some new modular relations for a remarkable product of theta–functions

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Abstract

In this paper, we establish some new modular equations of degree 9. We also establish several new $P\{Q$ mixed modular equations involving theta–functions which are similar to those recorded by Ramanujan in his notebooks. As an application, we establish some new general formulas for explicit evaluations of a Remarkable product of theta–functions.

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1 Introduction

For $|q| < 1$,

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}(q^2; q^2)_{\infty},$$  \hspace{1cm} (1.1)

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = (q^2; q^2)_{\infty} \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}},$$  \hspace{1cm} (1.2)

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty},$$  \hspace{1cm} (1.3)

are special cases of Ramanujan’s general theta function [4]

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ |ab| < 1,$$

$$= (a; q)_{\infty} (-b; q)_{\infty} (ab; q)_{\infty},$$

where $(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n)$.

At scattered places of his second notebook [18], Ramanujan records a total of nine $P\{Q$ mixed modular equations of degrees 1, 3, 5 and 15. These equations were proved by B. C. Berndt and L. -C. Zhang [6] and [7]. S. Bhargava, C. Adiga and M. S. Mahadeva Naika [9] and [10], have established several new $P\{Q$ modular equations involving four moduli. For more details on $P\{Q$ eta-function identities one can refer [1], [3], [12], [16] and [14].
In §2, we collect some identities that are needed to prove our main results. In §3, we establish some new modular equations of degree 9. In §4, we establish several new $P$–$Q$ mixed modular equations akin to those recorded by Ramanujan in his notebooks.

Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy [17], have introduced a new remarkable product of theta-functions $b_{s,t}$:

$$b_{s,t} = \frac{te^{-\frac{t-1-s}{4}\sqrt{\tau}}\psi^2\left(-e^{-\pi\sqrt{\tau}}\right)\varphi^2\left(-e^{-2\pi\sqrt{\tau}}\right)}{\psi^2\left(-e^{-\pi\sqrt{\tau}}\right)\varphi^2\left(-e^{-2\pi\sqrt{\tau}}\right)},$$  \hspace{1cm} (1.4)

where $s$, $t$ are real numbers such that $s > 0$ and $t \geq 1$. They have established some new general formulas for the explicit evaluations of $b_{s,t}$ and computed some particular values of $b_{s,t}$. Finally in §5, we establish some new modular relations connecting a remarkable product of theta-functions $b_{s,9}$ with $b_{r,s,9}$ for $r = 2, 4$ and $6$ and explicit values of $b_{s,9}$ are deduced.

We end this section by defining a modular equation in brief. The ordinary or Gaussian hypergeometric function is defined as

$$2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \hspace{0.5cm} 0 \leq |z| < 1,$$

and $a, b, c$ are complex numbers such that $c \neq 0, -1, -2, \ldots$,

where 

$$(a)_0 = 1, \hspace{0.5cm} (a)_n = a(a+1) \cdots (a+n-1) \hspace{0.5cm} \text{for} \hspace{0.5cm} n \hspace{0.5cm} \text{a positive integer}.$$

Let

$$K(k) := \int_0^{\pi} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^2}{(n!)^2} k^{2n} = \frac{\pi}{2} 2F_1\left(1, 1; 1; k^2\right),$$  \hspace{1cm} (1.5)

where $0 < k < 1$. The number $k$ is called the modulus of $K$, and $k' := \sqrt{1-k^2}$ is called the complementary modulus.

Let $K$, $K'$, $L$ and $L'$ denote the complete elliptic integrals of the first kind associated with the moduli $k$, $k'$, $l$ and $l'$, respectively.

Suppose that the equality

$$nK' = \frac{L'}{L},$$  \hspace{1cm} (1.6)

holds for some positive integer $n$. Then a modular equation of degree $n$ is a relation between the moduli $k$ and $l$ which is induced by (1.6). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say $\beta$ is of degree $n$ over $\alpha$. The multiplier $m$ is defined by

$$m = \frac{K}{L}. \hspace{1cm} (1.7)$$

Let $K, K', L_1, L'_1, L_2, L'_2, L_3$ and $L'_3$ denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli, respectively.

Let $n_1$, $n_2$ and $n_3$ be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L'_1}{L_1}, \hspace{0.5cm} n_2 \frac{K'}{K} = \frac{L'_2}{L_2} \hspace{0.5cm} \text{and} \hspace{0.5cm} n_3 \frac{K'}{K} = \frac{L'_3}{L_3},$$  \hspace{1cm} (1.8)
hold. Then a “mixed” modular equation is a relation between the moduli \( \sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma} \) and \( \sqrt{\delta} \) that is induced by (1.8). We say that \( \beta, \gamma \) and \( \delta \) are of degrees \( n_1, n_2 \) and \( n_3 \), respectively over \( \alpha \). The multipliers \( m \) and \( m' \) are associated with \( \alpha, \beta \) and \( \gamma, \delta \).

## 2 Preliminary results

In this section, we list some relevant identities which are useful in establishing our main results.

**Lemma 2.1.** [4, Ch. 17, Entry 12 (i) and (iii), p. 124] For \( 0 < x < 1 \), let

\[
\begin{align*}
    f(e^{-y}) &= \sqrt{2}z^{1/6}(x(1-x)e^y)^{1/24}, \\
    f(-e^{-2y}) &= \sqrt{2}z^{1/3}(x(1-x)e^{2y})^{1/12},
\end{align*}
\]

where \( z := \frac{2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}{2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} \).

**Lemma 2.2.** [4, Ch. 16, Entry 24 (ii) and (iv), p. 39] We have

\[
\begin{align*}
    f^3(q) &= \varphi^2(-q)\psi(q), \\
    f^3(-q^2) &= \varphi(-q)\psi^2(q).
\end{align*}
\]

**Lemma 2.3.** [4, Ch. 20, Entry 3 (x) and (xi), p. 352] Let \( \phi \) be of degree nine over \( \alpha \) and \( m \) be the multiplier relating \( \alpha \) and \( \beta \), then

\[
\begin{align*}
    \left( \frac{\beta}{\alpha} \right)^{1/8} + \left( \frac{1 - \beta}{1 - \alpha} \right)^{1/8} - \left( \frac{\beta(1 - \beta)}{\alpha(1 - \alpha)} \right)^{1/8} &= \sqrt{m}, \\
    \left( \frac{\alpha}{\beta} \right)^{1/8} + \left( \frac{1 - \alpha}{1 - \beta} \right)^{1/8} - \left( \frac{\alpha(1 - \alpha)}{\beta(1 - \beta)} \right)^{1/8} &= \frac{3}{\sqrt{m}}.
\end{align*}
\]

**Lemma 2.4.** [5, Ch. 25, Entry 56, p. 210] If \( P = \frac{f(-q)}{q^{1/3}f(-q^9)} \) and \( Q = \frac{f(-q^2)}{q^{2/3}f(-q^{18})} \), then

\[
P^3 + Q^3 = P^2Q^2 + 3PQ.
\]

**Lemma 2.5.** [3] If \( P = \frac{f(-q)}{q^{1/3}f(-q^9)} \) and \( Q = \frac{f(-q^3)}{qf(-q^{27})} \), then

\[
(PQ)^3 + \left( \frac{9}{PQ} \right)^3 + 27\left[ \left( \frac{P}{Q} \right)^3 + \left( \frac{Q}{P} \right)^3 \right] + 243\left( \frac{1}{P^3} + \frac{1}{Q^3} \right)
\]

\[+ 9(P^3 + Q^3) + 81 = \left( \frac{Q}{P} \right)^6.
\]

**Lemma 2.6.** [2] If \( P = \frac{\psi(-q)}{q\psi(-q^9)} \) and \( Q = \frac{\varphi(q)}{\varphi(q^9)} \), then

\[
Q + PQ = 3 + P.
\]
Lemma 2.7. If \( U := \frac{\phi(-q)}{\varphi(-q^9)} \) and \( V := \frac{\phi(-q^2)}{\varphi(-q^{18})} \), then
\[
\frac{U}{V} + \frac{V}{U} + 2 = V + \frac{3}{V}.
\] (2.10)

3 Modular equations
In this section, we establish some new modular equations of degree 9.

Theorem 3.1. If \( U := \frac{\phi(-q)}{\varphi(-q^9)} \) and \( V := \frac{\phi(-q^4)}{\varphi(-q^{36})} \), then
\[
4\left(U + \frac{2}{U}\right)\left(V + \frac{2}{V}\right) + 3\left(V^2 + \frac{9}{V^2}\right) + \frac{U^2}{V^2} + \frac{V^2}{U^2} + \frac{20}{UV} + 24 = \left(U + \frac{3}{U}\right)\left[6 + \left(V^2 + \frac{9}{V^2}\right)\right] + 12\left(V + \frac{3}{V}\right). \] (3.1)

Proof. Replacing \( q \) by \( q^2 \) in the equation (2.7), we deduce that
\[
Y^3 + X^3 = X^2Y^2 + 3XY, \] (3.2)
where
\[
X := \frac{f(-q^2)}{q^{2/3}f(-q^{18})} \quad \text{and} \quad Y := \frac{f(-q^4)}{q^{4/3}f(-q^{36})}.
\]
Equation (3.2) can be rewritten as
\[
a^2 + 3a - W = 0, \] (3.3)
where \( a := XY \) and \( W := Y^3 + X^3 \).
Solving the equation (3.3) for \( a \) and then cubing both sides, we deduce that
\[
27X^3Y^3 - X^9 + 6X^6Y^3 + 6X^3Y^6 - Y^9 + X^6Y^6 = 0. \] (3.4)
Using the equations (2.3), (2.4) and (2.9) in the equation (3.4), we find that
\[
(U^2 - 2U^2V + U^2V^2 - 6U + 9 + 4UV - 6V - 2UV^2 + V^2)\left(-U^2 + U^2V + 3U - 4UV + 3V - V^2 + UV^2\right) \quad (U^4 - U^3V + 4V^3U - 6V^2U^3 + 8VU^3 - 9U^3 + 3U^2V^4 - 12U^2V^3 + 24U^2V^2 - 36UV^2 + 27U^2 - 3UV^4 + 8UV^3 - 18UV^2 + 36UV - 27U + V^4) = 0
\] (3.5)
By examining the behavior of the above factors of the equation as \( q \to 0 \), we can find a neighborhood about the origin, where the last factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem the last factor vanishes identically. This completes the proof.
Theorem 3.2. If \( U := \frac{\varphi(-q)}{\varphi(-q^9)} \) and \( V := \frac{\varphi(-q^6)}{\varphi(-q^{54})} \), then

\[
18 \left(2U^2 + \frac{13}{U^2}\right) - 18 \left(4U + \frac{16}{U}\right) - 24V^2 \left(27U - \frac{1}{U}\right) - \frac{27}{V^3} \left(2U^2 + \frac{9}{U^2}\right)
- \left(V^3 + \frac{243}{V^3}\right) + \left(4V^2 + \frac{243}{V^2}\right) + \left(V + \frac{15}{V}\right) \left[15U - 6U^2 + U^3\right] + \frac{108U^2}{V^2}
+ \frac{3}{U} \left(V^3 + \frac{108}{V^3}\right) - 3U^3 \left(2 + \frac{6}{U^2} - \frac{3}{V^3}\right) - V^3 \left(\frac{3}{U^2} + \frac{1}{U^3}\right) + 186
+ 135U \left(\frac{1}{V^3} - \frac{2}{V^2}\right) = 0.
\]

Proof. The proof of the equation (3.6) is similar to the proof of the equation (3.1); except that in place of the equation (2.7), we use the equation (2.8).

Q.E.D.

4 Mixed modular equations

In this section, we establish several new mixed modular equations akin to those recorded by Ramanujan in his notebooks. Throughout this section, we set

\[
A := \frac{f(-q)f(-q^2)}{qf(-q^9)f(-q^{18})}, \quad B_n := \frac{f(-q^n)f(-q^{2n})}{q^n f(-q^{9n})f(-q^{18n})}
\]

and \( C_n := \frac{q^{n/3}f(-q^n)f(-q^{18n})}{f(-q^{2n})f(-q^{9n})} \).

Theorem 4.1. If \( U \neq 1 \) and \( V \neq 1 \), then

\[
\frac{f(-q)f(-q^2)}{qf(-q^9)f(-q^{18})} = \frac{U(3 - U)}{(1 - U)}, \quad \frac{f(-q)f(-q^2)}{qf(-q^9)f(-q^{18})} = \frac{V(3 - V)}{(1 - V)}.
\]

Theorem 4.2. If \( V \neq 3 \), then

\[
\frac{qf^3(-q)f^3(-q^{18})}{f^3(-q^2)f^3(-q^9)} = \frac{U(1 - U)}{(3 - U)}, \quad \frac{qf^3(-q)f^3(-q^{18})}{f^3(-q^2)f^3(-q^9)} = \frac{(1 - V)}{V(3 - V)}.
\]

where \( U := \frac{\varphi(-q)}{\varphi(-q^9)} \) and \( V := \frac{\psi(q)}{q\psi(q^9)} \).

Proof of (4.1) and (4.2). Using the equations (2.5) and (2.6), we find that

\[
\sqrt{m} \left\{ \frac{\alpha(1 - \alpha)}{\beta(1 - \beta)} \right\}^{1/8} + 1 = 3 \sqrt{m} + \left\{ \frac{\alpha(1 - \alpha)}{\beta(1 - \beta)} \right\}^{1/8}.
\]

(4.5)
Employing the equations (2.1) and (2.2) in the equation (4.5) and then changing \( q \) to \(-q\), we arrive at the equation (4.1). By using the equation (2.9) in the equation (4.1), we arrive at the equation (4.2).

*Proofs of (4.3) and (4.4).* The proofs of the equations (4.3) and (4.4) are similar to the proofs of the equations (4.1) and (4.2), respectively. Hence, we omit the details. \( \text{q.e.d.} \)

**Theorem 4.3.** If \( P := AB_2^2 \) and \( Q := \frac{A}{B_2} \), then

\[
Q^2 + \frac{1}{Q^2} = Q + \frac{1}{Q} + \left( \sqrt{P} + \frac{9}{\sqrt{P}} \right) \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) + 6. \quad (4.6)
\]

*Proof.* Using equations (2.3) and (2.9) in the equation (2.7), we deduce that

\[
\frac{\varphi^2(-q)}{\varphi^2(-q^9)} \left( \frac{\varphi(-q)}{\varphi(-q^9)} - 3 \right) + \frac{\varphi^2(-q^2)}{\varphi^2(-q^{18})} \left( \frac{\varphi(-q^2)}{\varphi(-q^{18})} - 3 \right) = A^2 + 3A. \quad (4.7)
\]

Using the equation (4.1) in the equation (4.7), we obtain

\[
4A^2 - 12A - 24B_2 - 12B_2^2 - 8B_2v + 8B_2vA + 4B_2^2vA - 4Au + 8B_2v \\
- 12B_2^2A + 12B_2^2u - 36B_2A + 24B_2u + 12vA + 4B_2^2v + 4B_2A^2 - 4B_2^3A \\
- 4vA^2 - 4B_2^3 - 4AuB_2 + 4Au - 4B_2^2vu + 4B_2^3u = 0, \quad (4.8)
\]

where \( u := \pm \sqrt{A^2 + 2A + 9} \) and \( v := \pm \sqrt{B_2^2 + 2B_2 + 9} \).

Collecting the terms containing \( u \) on one side of the equation (4.8) and then squaring both sides, we deduce that

\[
vA^2 - 5A^2 + B_2vA^2 - B_2^3A^2 - 2B_2A^2 - B_2^3vA - 3B_2vA + 3B_2^3A \\
- 2B_2^3vA + B_2^4A + 9B_2^2A + 7B_2A + 36B_2 - 12B_2v - 12B_2^3v + 6B_2^4 \\
- 5B_2^3v + 21B_2^3 + 40B_2^2 - B_2^4v + B_2^5 = 0. \quad (4.9)
\]

Eliminating \( v \) from the equation (4.9) and then setting then \( P := AB_2 \) and \( Q := \frac{A}{B_2} \), we arrive at the equation (4.6). \( \text{q.e.d.} \)
Theorem 4.4. If $P := AB_4$ and $Q := \frac{A}{B_4}$, then

$$
\left( \sqrt{P} + \frac{3}{\sqrt{P}} \right) \left[ -170 \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) - 84 \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) - 13 \left( \sqrt{Q^5} + \frac{1}{\sqrt{Q^5}} \right) \right]
$$

$$
+ \left( P + \frac{3^4}{P} \right) \left[ -31 - 4 \left( Q^2 + \frac{1}{Q^2} \right) - 34 \left( Q + \frac{1}{Q} \right) \right] + \left( \sqrt{P^3} + \frac{3^6}{\sqrt{P^3}} \right)
$$

$$
\times \left[ -6 \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) - \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) \right] - \left( P^2 + \frac{3^8}{P^2} \right) - 365 \left( Q + \frac{1}{Q} \right)
$$

$$
- 176 \left( Q^2 + \frac{1}{Q^2} \right) - 14 \left( Q^3 + \frac{1}{Q^3} \right) + \left( Q^4 + \frac{1}{Q^4} \right) = 782.
$$

(4.10)

Proof. Using the equation (4.1) in the equation (3.1), we deduce that

$$
2592 - 1440A - 2304B_4 - 864v + 864u - 736uB_4 - 96A^2uB_4^3v - 32A^2uB_4^3v
$$

$$
+ 96A^3B_4^2v + 32A^3B_4^2v + 480A^2B_4^2v + 288A^2B_4^2v + 96A^2B_4^2v + 224A^3B_4v
$$

$$
+ 512A^2uB_4 - 32A^2uB_4^2 + 448A^2uB_4^2 + 128A^2uB_4^2 + 32A^2uB_4^2 - 64AuB_4^3v
$$

$$
+ 1888AB_4v + 992AB_4^3v + 288AB_4^3v + 640AuB_4 + 128AuB_4 + 64A^4 + 608A^2
$$

$$
+ 704AuB_4^2 + 256AuB_4^3 + 64AuB_4^4 - 416uB_4^2 - 96uB_4^2v - 192AuB_4^2v
$$

$$
- 576Au + 224A^2u - 2560B_4^2 - 864B_4v + 800B_4^2v + 224B_4^3v - 224B_4^2v - 64AuB_4v
$$

$$
- 512A^3B_4 - 448A^3B_4^2 - 128A^3B_4^2 - 32A^3B_4^2 - 1152A^2B_4 + 288Au - 64A^3u
$$

$$
- 96A^2v - 1152A^2B_4^2 - 384A^2B_4^2 - 96A^2B_4^2 - 4992AB_4 - 256AuB_4v
$$

$$
- 4032AB_4^2 - 1280AB_4^2 - 288AB_4^3 + 2304uB_4 - 288uv + 1536uB_4^2 + 32A^3v
$$

$$
+ 512uB_4^3 + 96uB_4^4 - 160A^3 - 1024B_4^3 - 224B_4^3 = 0,
$$

where $u := \pm \sqrt{A^2 + 2A + 9}$ and $v := \pm \sqrt{B_4^2 + 2B_4 + 9}$.

Collecting the terms containing $u$ on one side of the equation (4.11) and then squaring both sides, we deduce that

$$
- 216AB_4v - 68A^2B_4^3v - 114A^2B_4^3v - 81A^2B_4^3v - 11A^3B_4^3v - 28A^2B_4^5v
$$

$$
- 288AB_4^3v - 181AB_4^3v - A^2B_4^3v - A^2B_4^3v - 2AB_4^3v + 1458B_4 - 15AB_4^3v
$$

$$
+ 2430B_4^3 + 1998B_4^3 - A^4 + 1034B_4^4 + 362B_4^5 + 86B_4^6 + 13B_4^7 + B_4^8 - 6A^2B_4^5v
$$

$$
- 486A_4v - 756B_4^3v - 558B_4^3v + 34A^3B_4 + 33A^3B_4^2 + 19A^3B_4^3 - 68AB_4^3v
$$

$$
+ 5A^5B_4^4 + 261A^2B_4^4 + 351A^2B_4^4 + 266A^2B_4^4 + 116A^2B_4^4 + 2AB_4^7 - 10A^3B_4v
$$

$$
+ 648AB_4 + 936AB_4^3 + 673AB_4^3 + 301AB_4^4 - 248B_4^4v + 38A^2B_4^5 - 4A^3B_4^3v
$$

$$
+ 91AB_4^5 - A^4B_4 + A^4v - 70B_4v + 7B_4^6A + 17B_4^6A - 12B_4^6v
$$

$$
- B_4^7v + A^2B_4^5 + A^2B_4^8 = 0.
$$

(4.12)

Eliminating $v$ from the equation (4.12) and then setting $P := AB_4$ and $Q := \frac{A}{B_4}$, we arrive at the equation (4.10).

Q.E.D.
Theorem 4.5. If $P := AB_6$ and $Q := \frac{A}{B_6}$, then

\[
\left(\sqrt{P} + \frac{3^2}{\sqrt{P}}\right) \left[ 162 \left( 846\sqrt{Q} + \frac{1}{\sqrt{Q}} \right) + 243 \left( 2610\sqrt{Q^3} - \frac{178}{\sqrt{Q^3}} \right) \\
+ 243 \left( 1215\sqrt{Q^5} - \frac{65}{\sqrt{Q^5}} \right) + 3 \left( 116640\sqrt{Q^7} - \frac{763}{\sqrt{Q^7}} \right) + 8748\sqrt{Q^{11}} \\
+ 6 \left( 14580\sqrt{Q^9} - \frac{17}{\sqrt{Q^9}} \right) \right] + (P + \frac{3^4}{P}) \left[ 23328 + 81 \left( 1554Q - \frac{137}{Q} \right) \\
+ 27 \left( 6786Q^2 - \frac{286}{Q^2} \right) + 729 \left( 190Q^3 - \frac{2}{Q^3} \right) + 46656Q^4 - \frac{61}{Q^4} + 7290Q^5 \right] \\
+ \left( \sqrt{P^3} + \frac{3^6}{\sqrt{P^3}} \right) \left[ 243 \left( 60\sqrt{Q} - \frac{4}{\sqrt{Q}} \right) + 9 \left( 4164\sqrt{Q^3} - \frac{299}{\sqrt{Q^3}} \right) \\
+ 648 \left( 60\sqrt{Q^5} - \frac{1}{\sqrt{Q^5}} \right) + 27 \left( 684\sqrt{Q^7} - \frac{1}{\sqrt{Q^7}} \right) + 3888\sqrt{Q^9} \right] \\
+ \left( P^2 + \frac{3^8}{P^2} \right) \left[ 729 + 27 \left( 176Q - \frac{36}{Q} \right) + 27 \left( 300Q^2 - \frac{8}{Q^2} \right) + 1539Q^4 \right] \\
+ 6 \left( 864Q^3 - \frac{1}{Q^3} \right) \right] + \left( \sqrt[3]{P^3} + \frac{3^{10}}{\sqrt[3]{P^3}} \right) \left[ 38\sqrt{Q} + \frac{13}{\sqrt{Q}} \right] + 532\sqrt{Q^7} \\
+ \left( 270\sqrt{Q^5} + \frac{54}{\sqrt{Q^5}} \right) + \left( 1080\sqrt{Q^7} + \frac{27}{\sqrt{Q^7}} \right) \right] + 9 \left( 186057Q^3 - \frac{2518}{Q^3} \right) \\
+ \left( P^3 + \frac{3^{12}}{P^3} \right) \left[ -13 + \left( 90Q - \frac{9}{Q} \right) + 144Q^2 \right] + 4374 \left( 351Q^2 - \frac{22}{Q^2} \right) \\
+ \left( \sqrt[5]{P^5} + \frac{3^{14}}{\sqrt[5]{P^5}} \right) \left[ \frac{1}{\sqrt{Q}} + 12\sqrt{Q^3} + 12\sqrt{Q^5} \right] + \left( P^4 + \frac{3^{16}}{P^4} \right) (Q^2) \\
+ 972 \left( 1539Q - \frac{143}{Q} \right) + 81 \left( 8100Q^4 - \frac{277}{Q^4} \right) + 9 \left( 11664Q^5 - \frac{7}{Q^5} \right) \\
+ \left( 6561Q^6 + \frac{1}{Q^6} \right) + 373491 = 0.
\]

Proof. The proof of the equation (4.13) is similar to the proof of the equation (4.10). So we omit the details. \(\text{q.e.d.}\)

Theorem 4.6. If $P := C_1C_2$ and $Q := \frac{C_1}{C_2}$, then

\[
\left( \sqrt{P} + \frac{1}{\sqrt{P}} \right) \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) = \left( P + \frac{1}{P} \right). \quad (4.14)
\]
Proof. Using the equation (4.3) in the equation (2.10), we deduce that
\[
3u - 3v - 2C_1^3 u - C_2^3 v + 6C_1^3 C_2^3 - C_3^3 v + u C_2^6 - 6u C_1^3 + u v - u C_2^3 v \\
+ C_1^3 C_2^3 v + 2C_2^6 - 1 + C_1^6 - C_1^3 C_2^3 - 13C_1^3 - 2C_2^3 = 0,
\]
where \(u := \pm \sqrt{C_1^2 - 10C_1 + 1}\) and \(v := \pm \sqrt{C_2^2 - 10C_2 + 1}\).

Collecting the terms containing \(u\) on one side of the equation (4.15) and then squaring both sides, we deduce that
\[
2C_2^3 v - 33C_1^3 C_2^3 + 9C_1^3 v + 40C_1^3 C_2^6 + 2C_1^6 - 7C_2^6 + 9C_1^3 - 2C_2^3 - 12C_1^3 C_2^3 v \\
+ C_1^6 C_2^3 v - C_2^9 C_3^3 + 8C_2^6 C_1^3 - 13C_2^6 C_1^3 - C_2^6 C_1^3 + C_2^6 C_2^3 + C_2^9 \\
- 2v C_2^6 + 7C_1^6 C_2^3 = 0.
\]

Collecting the terms containing \(v\) on one side of the equation (4.16) and then squaring both sides, we arrive at
\[
A(q)B(q) = 0,
\]
where \(A(q) = C_2 C_1^4 + C_1^3 - C_1 C_2 - C_1^3 C_2^2 + C_3^3 + C_1^4 C_1\) and
\[
B(q) = C_2^3 C_1^8 + C_1^3 C_2^7 - C_2 C_1^7 + C_1^6 C_2^6 - 2C_1^6 C_2^3 + 2C_2^5 C_1^5 - 2C_2 C_1^5 \\
+ C_2^7 C_1^4 - 3C_4^5 C_1^4 + C_2 C_1^4 - 2C_1^3 C_2^6 + 2C_1^3 C_2^3 + C_1^6 + C_2^8 C_1^4 \\
- 2C_2^5 C_1^2 + C_2^5 C_1^2 - C_2^7 C_1 + C_2^4 C_1 + C_2^6.
\]

Consider the sequence \(\{q_n\} = \left\{ \frac{1}{n+1} \right\}\), \(n = 1, 2, 3\ldots\), which has a limit in \(|q| < 1\). We see \(A(q_n) = 0\) \(\forall n\), where as \(B(q_n) \neq 0\) \(\forall n\). Then by zeros of analytic functions, \(A \equiv 0\) on \(|q| < 1\) \([8]\).

By setting \(P := C_1 C_2\) and \(Q := \frac{C_1}{C_2}\), we arrive at (4.14). This completes the proof. \(\text{q.e.d.}\)

Theorem 4.7. If \(P := C_1 C_4\) and \(Q := \frac{C_1}{C_4}\), then
\[
\left(\sqrt{P^3} + \frac{1}{\sqrt{P^3}}\right) \left[ 5 \left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + 4 \left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + \left(\sqrt{Q^5} + \frac{1}{\sqrt{Q^5}}\right) \right] \\
+ 14 \left(\sqrt{Q} + \frac{1}{Q}\right) + 7 \left(\sqrt{Q^2} + \frac{1}{\sqrt{Q^2}}\right) + \left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) + 12 = \left(P^3 + \frac{1}{P^3}\right).
\]

Proof. The proof of the equation (4.19) is similar to the proof of the equation (4.14); except that in place of the equation (2.10), we use the equation (3.1). \(\text{q.e.d.}\)

5 A remarkable product of theta–functions

Mahadeva Naika, Maheshkumar and Sushan Bairy [17], have introduced a new remarkable product of theta–functions as in the equation (1.4). They have also established some new general formulas for the explicit evaluations of \(b_{s,t}\).
Recently, Mahadeva Naika, Chandankumar and Hemanthkumar[13], have established several new modular identities connecting the remarkable product of theta–functions \( b_{s,9} \) with \( b_{r^2 s,9} \) for \( r = 3, 5 \) and \( 11 \) and also established some new values for \( b_{s,9} \).

In this section, we establish several new modular identities connecting the remarkable product of theta–functions \( b_{s,9} \) with \( b_{r^2 s,9} \) for \( r = 2, 4 \) and \( 6 \).

**Lemma 5.1.** [17] If \( s \) and \( t \) are any positive rational, then
\[
b_{2s,t}b_{2s,t} = 1.
\]

**Lemma 5.2.** [15] We have, \( 0 < b_{s,t} \leq 1 \) for all \( s \geq 2 \) and \( t \) positive integer greater than 1.

**Theorem 5.3.** If \( X := \sqrt{b_{s,9}b_{4s,9}} \) and \( Y := \sqrt{\frac{b_{4s,9}}{b_{s,9}}} \), then
\[
Y^2 + \frac{1}{Y^2} = Y + \frac{1}{Y} + 3 \left( \sqrt{X} + \frac{1}{\sqrt{X}} \right) \left( \sqrt{Y} + \frac{1}{\sqrt{Y}} \right) + 6.
\]

**Proof.** Using the equation \((4.6)\) along with the equation \((1.4)\) with \( t := 9 \), we arrive at \((5.2)\). Q.E.D.

**Corollary 5.4.** We have
\[
b_{1,9} = \left( \frac{3}{2} + \frac{3^{1/4}}{\sqrt{2}} \right)^2,
\]
\[
b_{4,9} = \left( \frac{3}{2} - \frac{3^{1/4}}{\sqrt{2}} \right)^2.
\]

**Proofs of \((5.3)\) and \((5.4)\).** Putting \( s = 1 \) in the equation \((5.2)\) and using the fact that \( b_{1,9}b_{4,9} = 1 \), we deduce that
\[
(h^4 - 2h^3 - 2h + 1)(h^2 + h + 1)^2 = 0,
\]
where \( h := \sqrt{b_{4,9}} \). We observe that the first factor of the equation \((5.5)\) vanishes and other factors does not vanish for the specific value of \( q = e^{-\pi \sqrt{4/9}} \). Hence, we have
\[
u^2 - 2u - 2 = 0,
\]
where \( u := \sqrt{b_{4,9}} + \frac{1}{\sqrt{b_{4,9}}} \).

On solving the equation \((5.6)\) and \( u > 0 \) by Lemma \((5.2)\), we find that
\[
\sqrt{b_{4,9}} + \frac{1}{\sqrt{b_{4,9}}} = 1 + \sqrt{3}.
\]
On solving the equation \((5.7)\), we arrive at \((5.3)\) and \((5.4)\). Q.E.D.

**Remark 5.5.** Another proof of \( b_{1,9} \) and \( b_{4,9} \) can be found in [13].
Theorem 5.6. If \( X := \sqrt{b_{8,9} b_{16,9}} \) and \( Y := \sqrt{\frac{b_{16,9}}{b_{8,9}}} \), then

\[
3 \left( \sqrt{X} \frac{1}{\sqrt{X}} \right) \left[ -170 \left( \sqrt{Y} \frac{1}{\sqrt{Y}} \right) - 84 \left( \sqrt{Y^3} \frac{1}{\sqrt{Y^3}} \right) - 13 \left( \sqrt{Y^5} \frac{1}{\sqrt{Y^5}} \right) \right] \\
+ 9 \left( X + \frac{1}{X} \right) \left[ -31 - 4 \left( Y^2 + \frac{1}{Y^2} \right) - 34 \left( Y + \frac{1}{Y} \right) \right] + 27 \left( \sqrt{X^3} \frac{1}{\sqrt{X^3}} \right) \\
\times \left[ -6 \left( \sqrt{Y} \frac{1}{\sqrt{Y}} \right) - \left( \sqrt{Y^3} + \frac{1}{\sqrt{Y^3}} \right) \right] - 81 \left( X^2 + \frac{1}{X^2} \right) - 365 \left( Y + \frac{1}{Y} \right) \\
- 176 \left( Y^2 + \frac{1}{Y^2} \right) - 14 \left( Y^3 + \frac{1}{Y^3} \right) + \left( Y^4 + \frac{1}{Y^4} \right) = 782. \tag{5.8}
\]

Proof. Using the equation (4.10) along with the equation (1.4) with \( t := 9 \), we arrive at (5.8). Q.E.D.

Corollary 5.7. We have

\[
b_{8,9} = \frac{(\sqrt{3} - \sqrt{2})^2(\sqrt{3} - 1)^2}{2}, \tag{5.9}
\]

\[
b_{1/2,9} = \frac{(\sqrt{3} + \sqrt{2})^2(\sqrt{3} + 1)^2}{2}. \tag{5.10}
\]

Proofs of (5.9) and (5.10). Putting \( s = 1/2 \) in the equation (5.8) and using the fact that \( b_{1/2,9} b_{8,9} = 1 \), we deduce

\[
(h^4 - 4h^3 - 12h^2 - 4h + 1) \left( h^2 + h + 1 \right)^2 (h^4 + h^3 + 3h^2 + h + 1)^2 = 0, \tag{5.11}
\]

where \( h := \sqrt{b_{8,9}} \).

We observe that the first factor of the equation (5.11) vanishes and other factors does not vanish for the specific value of \( q = e^{-\pi \sqrt{8/9}} \). Hence, we have

\[
u^2 - 4u - 14 = 0, \tag{5.12}
\]

where \( u := \sqrt{b_{8,9}} + \frac{1}{\sqrt{b_{8,9}}} \).

On solving the equation (5.12) and \( u > 0 \), we find that

\[
\sqrt{b_{8,9}} + \frac{1}{\sqrt{b_{8,9}}} = 2 + 3\sqrt{2}. \tag{5.13}
\]

On solving the equation (5.13), we arrive at (5.9) and (5.10). Q.E.D.
Theorem 5.8. If \( X := \sqrt{b_{s,9}b_{36s,9}} \) and \( Y := \sqrt{\frac{b_{36s,9}}{b_{s,9}}} \), then

\[
3 \left( \sqrt{X} + \frac{1}{\sqrt{X}} \right) \left[ 162 \left( 846\sqrt{Y} + \frac{1}{\sqrt{Y}} \right) + 243 \left( 2610\sqrt{Y^3} - \frac{178}{\sqrt{Y^3}} \right) \\
+ 1215\sqrt{Y^5} - \frac{65}{\sqrt{Y^5}} \right] + 3 \left( 116640\sqrt{Y^7} - \frac{763}{\sqrt{Y^7}} \right) + 8748\sqrt{Y^{11}} \\
+ 6 \left( 14580\sqrt{Y^9} - \frac{17}{\sqrt{Y^9}} \right) + 9 \left( X + \frac{1}{X} \right) \left[ 23328 + 81 \left( 1554Y - \frac{137}{Y} \right) \\
+ 6786Y^2 - \frac{286}{Y^2} \right] + 729 \left( 190Y^3 - \frac{2}{Y^3} \right) + \left( 46656Y^4 - \frac{61}{Y^4} \right) + 7290Y^5 \\
+ 27 \left( \sqrt{X^3} + \frac{1}{\sqrt{X^3}} \right) \left[ 243 \left( 60\sqrt{Y} - \frac{4}{\sqrt{Y}} \right) + 9 \left( 4164\sqrt{Y^3} - \frac{299}{\sqrt{Y^3}} \right) \\
+ 648 \left( 60\sqrt{Y^5} - \frac{1}{\sqrt{Y^5}} \right) + 27 \left( 684\sqrt{Y^7} - \frac{1}{\sqrt{Y^7}} \right) + 3888\sqrt{Y^9} \right] \\
+ 81 \left( X^2 + \frac{1}{X^2} \right) \left[ 729 + 27 \left( 176Y - \frac{36}{Y} \right) + 27 \left( 300Y^2 - \frac{8}{Y^2} \right) + 1539Y^4 \right] \\
+ 6 \left( 864Y^3 - \frac{1}{Y^3} \right) + 3^5 \left( \sqrt{X^5} + \frac{1}{\sqrt{X^5}} \right) \left[ 38\sqrt{Y} + \frac{13}{\sqrt{Y}} \right] + 532\sqrt{Y^7} \\
+ \left( 270\sqrt{Y^3} + \frac{54}{\sqrt{Y^3}} \right) + \left( 1080\sqrt{Y^5} + \frac{27}{\sqrt{Y^5}} \right) + 9 \left( 186057Y^3 - \frac{2518}{Y^3} \right) \\
+ 3^6 \left( X^3 + \frac{1}{X^3} \right) \left[ -13 + \left( 90Y - \frac{9}{Y} \right) + 144Y^2 \right] + 4374 \left( 351Y^2 - \frac{22}{Y^2} \right) \\
+ 3^7 \left( \sqrt{X^7} + \frac{1}{\sqrt{X^7}} \right) \left[ \frac{1}{\sqrt{Y}} + 12\sqrt{Y^3} + 12\sqrt{Y^5} \right] + 3^8 \left( X^4 + \frac{1}{X^4} \right) \left( Y^2 \right) \\
+ 972 \left( 1539Y - \frac{143}{Y} \right) + 81 \left( 8100Y^4 - \frac{277}{Y^4} \right) + 9 \left( 11664Y^5 - \frac{7}{Y^5} \right) \\
+ \left( 6561Y^6 + \frac{1}{Y^6} \right) + 373491 = 0.
\]

Proof. Using the equation (4.13) along with the equation (1.4) with \( t := 9 \), we arrive at (5.14).

Q.E.D.

Corollary 5.9. We have

\[
b_{12,9} = \frac{\left( (5 - 3\sqrt{3})x^2 + x - 1 \right)^2}{9}, \tag{5.15}
\]

\[
b_{1/3,9} = \left( x^2 + x\sqrt{3} + 2 + \sqrt{3} \right)^2, \tag{5.16}
\]

where \( x := (5 + 3\sqrt{3})^{1/3} \).
Proofs of (5.15) and (5.16). Setting \( s = 1/3 \) in the equation (5.14) and using the fact that \( b_{1/3,9}b_{12,9} = 1 \), we deduce that

\[
\begin{align*}
(9h^6 + 18h^5 + 27h^4 + 6h^3 - 3h^2 - 12h + 1) & \left( 3h^3 + 3h^2 + 3h + 1 \right)^2 \\
(9h^6 + 18h^5 + 27h^4 + 21h^3 + 12h^2 + 3h + 1)^2 &= 0, \\
\end{align*}
\]  

(5.17)

where \( h := \sqrt{b_{12,9}} \).

We observe that the first factor of the equation (5.17) vanishes and other factors does not vanish for the specific value of \( q = e^{-\pi \sqrt{12/9}} \). Hence, we have

\[
(9h^6 + 18h^5 + 27h^4 + 6h^3 - 3h^2 - 12h + 1) = 0.
\]  

(5.18)

On solving the equation (5.18) by using Maple and \( 0 < h < 1 \), we arrive at (5.15) and (5.16). Q.E.D.

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