Penetrative ferroconvection via internal heating in a saturated porous layer with constant heat flux at the lower boundary

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A R T I C L E   I N F O

Article history:
Received 10 April 2011
Received in revised form 12 November 2011
Accepted 28 November 2011
Available online 14 December 2011

Keywords:
Penetrative ferroconvection
Porous layer
Internal heat generation
Viscosity ratio

A B S T R A C T

A model for penetrative ferroconvection via internal heat generation in a ferrofluid saturated porous layer is explored. The Brinkman–Lapwood extended Darcy equation with fluid viscosity different from effective viscosity is used to describe the flow in the porous medium. The lower boundary of the porous layer is assumed to be rigid-paramagnetic and insulated to temperature perturbations, while at upper stress-free boundary a general convective-radiative exchange condition on perturbed temperature is imposed. The resulting eigenvalue problem is solved numerically using the Galerkin method. It is found that increasing in the dimensionless heat source strength imposes numerically using the Galerkin method. It is found that increasing in the dimensionless heat source strength, magnetic number $M$, Biot number $Bi$, and magnetic susceptibility $\chi$ is to delay the onset of ferroconvection. Further, increase in $Bi$, $Da^{-1}$ and $N_s$, and decrease in $M$, $M_1$ and $M_2$ is to diminish the dimension of convection cells.

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1. Introduction

Ferrofluids are stable colloidal suspensions of magnetic nanoparticles in a carrier fluid such as water, hydrocarbon (mineral oil or kerosene), or fluorocarbon. The nanoparticles typically have sizes of about 100 Å or 10 nm and they are coated with surfactants in order to prevent the coagulation. Ferrofluids respond to an external magnetic field and this enables to control the location of the ferrofluid through the application of a magnetic field. Ferrofluids have been tailor-made and possess a wide variety of potential applications in various industries \cite{1–5}. Mechanical engineering industries use them as fluids in vibration dampers, shock absorbers, and vacuum seals. Electrical and electronic industries use ferrofluids to improve hi-fi characteristic loud speakers as transformer coolants and also in miniaturizing inductive components. Computation industries use them as fluids in stepper motors. Therefore, studies on ferrofluids have received much attention in the scientific community over the years.

The magnetization of ferrofluids depends on the magnetic field, temperature, and density. Hence, any variations in these quantities induce a change in body force distribution in the fluid and eventually give rise to convection in ferrofluids in the presence of a gradient of magnetic field. There have been numerous studies on thermal convection in a ferrofluid layer called ferroconvection analogous to Rayleigh–Benard convection in ordinary viscous fluids. Finlayson \cite{6} has studied convective instability of a magnetic fluid layer heated from below in the presence of a uniform vertical magnetic field. A linear stability analysis has been carried out to predict the critical gradient of temperature corresponding to the onset of convection when both buoyancy and magnetic forces are included, by considering the bounding surfaces of the magnetic fluid layer to be either stress-free or rigid. Thermo-convective instability of ferrofluids without considering buoyancy effects has been investigated by Lalas and Carmi \cite{7}, whereas Shliomis \cite{8} has analyzed the linear relation for magnetized perturbed quantities at the limit of instability. A similar analysis but with the fluid confined between ferromagnetic plates has been carried out by Gotoh and Yamada \cite{9} using linear stability analysis. Schwab et al. \cite{10} have experimentally investigated the problem of Finlayson in the case of a strong magnetic field and detected the onset of convection by plotting the Nusselt number versus the Rayleigh number. Stiles and Kagan \cite{11} have extended the problem to allow for the dependence of effective shear viscosity on temperature and colloid concentration. The effect of the different forms of basic temperature gradients on the onset of ferroconvection driven by combined surface tension and buoyancy forces has been discussed by Shivakumara et al. \cite{12} in order to understand the control of ferroconvection. Kaloni and Lou \cite{13} have theoretically investigated the convective instability problem in a thin horizontal layer.
of magnetic fluid heated from below under alternating magnetic field, by considering the quasi-stationary model with internal rotation and vortex viscosity. Shivakumara and Nanjundappa [14] have analyzed the effect of various forms of nonuniform initial temperature profiles on the onset of Marangoni convection in a ferrofluid layer. The influence of magnetic field on heat and mass transport in ferrofluids has been discussed by Volker et al. [15].

Sunil and Mahajan [16] have performed nonlinear stability analysis for a magnetized ferrofluid layer heated from below in the stress-free boundaries, while Nanjundappa and Shivakumara [17] have analyzed the effects of different velocities and temperature boundary conditions on the onset of convection in a ferrofluid layer. The effect of magnetic field dependent viscosity on the onset of thermal convection in a horizontal ferrofluid layer heated from below has been studied by Nanjundappa et al. [18]. Recently, Nanjundappa et al. [19] have investigated theoretically the effect of magnetic field dependent viscosity on the onset of Benard-Marangoni ferroconvection in a horizontal ferrofluid layer.

Thermal convection of ferrofluids saturating a porous medium has also attracted considerable attention in the literature, owing to its importance in controlled emplacement of liquids or treatment of chemicals, and emplacement of geophysically imageable liquids into particular zones for subsequent imaging. Rosensweig et al. [20] have experimentally studied the penetration of ferrofluids in the Hele-Shaw cell. The stability of the magnetic fluid penetration through a porous medium in high uniform magnetic field oblique to the interface is studied by Zahn and Rosensweig [21]. The thermal convection of a ferrofluid saturating a porous medium in the presence of a vertical magnetic field is studied by Vaidyanathan et al. [22] by employing the Brinkman equation with effective viscosity (Brinkman viscosity) is the same as fluid viscosity and considering that the bounding surfaces of the porous layer are shear free. Qin and Chadam [23] have carried out the nonlinear stability analysis of ferroconvection in a porous layer by including the inertial effects to accommodate high velocity. The laboratory-scale experimental results of the behavior of ferrofluids in porous media consisting of sands and sediments are presented by Borglin et al. [24]. The onset of centrifugal convection in a magnetic fluid-saturated porous medium under zero gravity condition is investigated by Saravanan and Yamaguchi [25]. The effect of dust particles on the onset of ferroconvection in a porous medium has been studied by Sunil et al. [26]. Shivakumara et al. [27] have investigated in detail the onset of ferroconvection in a ferrofluid saturated porous medium for various types of velocity and temperature boundary conditions. Nanjundappa et al. [28] have performed linear stability analysis to investigate buoyancy driven convection in a ferrofluid saturated porous medium. Recently, Shivakumara et al. [29] have studied the effect of Coriolis force on the onset of ferromagnetic convection in a rotating horizontal ferrofluid saturated porous layer in the presence of a uniform vertical magnetic field.

The practical problems cited above require a mechanism to control thermomagnetic convection. One of the mechanisms to control (suppress or augment) convection is by maintaining a nonuniform temperature gradient across the layer of ferrofluid. Such a temperature gradient may arise due to (i) uniform distribution of heat sources (ii) transient heating or cooling at a boundary, (iii) temperature modulation at the boundaries and so on. Works have been carried out in this direction but it is still in much-to-be-desired state. Rudraiah and Sekhar [30] have investigated convection in a ferrofluid layer in the presence of uniform internal heat source. The effect of non-uniform basic temperature gradients on the onset of ferroconvection has been analyzed by Shivakumara and Nanjundappa [12,31,32]. Singh and Bajaj [33] have studied thermal convection of ferrofluids in the presence of uniform vertical magnetic field with boundary temperatures modulated sinusoidally about some reference value. Idris and Hashim [34] have investigated the instability of Benard-Marangoni ferroconvection in a horizontal layer of ferrofluid under the influence of a linear feedback control and cubic temperature gradient. Recently, Nanjundappa et al. [35] have studied the effect of internal heat generation on the criterion for the onset of convection in a horizontal ferrofluid saturated porous layer in the presence of a uniform magnetic field.

The intent of the present study is to investigate ferroconvection in a ferrofluid-saturated porous layer in the presence of internal heating. The presence of internal heating deviates the basic temperature, magnetic field intensity and magnetization distributions from linear to parabolic with respect to porous layer height, which in turn play a decisive role in understanding control of ferroconvection. Besides, porous materials used in many technological applications of practical importance possess high permeability values. For example, permeabilities of compressed foams as high as 8 × 10−6 m² and for a 1 mm thick foam layer the equivalent Darcy number is equal to 8 (see Nield et al. [36] and references therein). For a high porosity porous medium (ϕ = 0.972), Givler and Alotbella [37] have determined experimentally that μt = 7.5×10⁻²4 Hs, where μt is the effective viscosity and μs is the fluid viscosity. Accordingly, the flow in the porous medium is described by the Brinkman–Lapwood extended Darcy equation with fluid viscosity different from effective or Brinkman viscosity. The resulting eigenvalue problem is solved by the Galerkin technique with modified Chebyshev polynomials as trial functions. The available results in the literature are obtained as limiting cases from the present study.

To achieve the above objectives, the remainder of this paper proceeds as follows. Section 2 is devoted to the mathematical formulation of the problem. The method of solution is discussed in Section 3. In Section 4, the numerical results are discussed and some important conclusions follow in Section 5.

2. Formulation of the Problem

The system considered is an initially quiescent incompressible constant viscosity ferrofluid saturated horizontal porous layer of characteristic thickness d in the presence of a uniform applied magnetic field H0 in the vertical direction (see Fig. 1). The horizontal extension of the porous layer is sufficiently large so that edge effects may be neglected. The boundaries are maintained at constant but different temperatures having higher temperature at the bottom T0 and lower temperature at the top (T0 − ΔT). In addition, a uniformly distributed overall internal heat source is present within the ferrofluid-saturated porous medium. A Cartesian co-ordinate system (x, y, z) is used with the origin at the bottom of the porous layer and z-axis is directed vertically upward. The flow in the porous medium is described by the Brinkman–Lapwood extended Darcy equation with fluid viscosity different from effective viscosity and the Oberbeck-Boussinesq approximation is assumed to be valid. Raja-gopal et al. [38] have presented a frame work within which the status of the Oberbeck-Boussinesq approximation can be clearly
delineated, within the context of a full thermodynamical theory for the Navier–Stokes fluid.

The governing equations for the flow of an incompressible magnetic fluid in a layer of porous medium are:

Continuity equation

$$\nabla \cdot \mathbf{V} = 0.$$  (1)

Momentum equation

$$\rho_0 \left[ \frac{1}{c_T} \frac{\partial \mathbf{V}}{\partial t} + \frac{1}{c_T} \nabla \cdot \mathbf{V} \right] = -\nabla p + \rho \mathbf{g} - \frac{\partial}{\partial t} \left( \rho_0 \mathbf{C} \right) + \frac{\partial}{\partial t} \left( \rho_0 \mathbf{C} \right) + \mu_0 (\mathbf{M} \cdot \nabla) \mathbf{H}.$$  (2)

Energy equation

$$\varepsilon = \rho_0 C_V, H - \mu_0 \mathbf{H} \cdot \left( \frac{\partial \mathbf{M}}{\partial t} \right) + \frac{\left( \frac{\partial \mathbf{M}}{\partial t} \right)}{\partial t} + (1-\varepsilon) \rho_0 C_v \frac{\partial T}{\partial t} + \mu_0 T \frac{\partial H}{\partial t} = k_1 \nabla^2 T + Q.$$  (3)

Equation of state

$$\rho = \rho_0 [1-\chi(T-T_0)].$$  (4)

Maxwell equations in the magnetostatic limit

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = 0 \quad \text{or} \quad \mathbf{H} = \nabla \varphi.$$  (5a, b)

$$\mathbf{B} = \mu_0 (\mathbf{M} + \mathbf{H}).$$  (6)

In the above equations, \( \mathbf{V} = (u, v, w) \) is the velocity of the fluid, \( \varepsilon \) the porosity of the porous medium, \( \mu_0 \) the dynamic viscosity, \( \mu \) the effective viscosity, \( p \) the pressure, \( \rho_0 \) the reference density, \( T \) the temperature, \( k \) the permeability of the porous medium, \( k_1 \) the overall thermal conductivity, \( \mu_0 \) the free space magnetic permeability, \( \alpha_t \) the thermal expansion co-efficient, \( C_v \) the specific heat at constant volume and magnetic field, \( B \) the magnetic induction field, \( H \) the magnetic field, \( \mu_0 \) the magnetic constant, \( H_b \) the constant magnetic field, \( M \) the magnetization, \( M_b \) the magnetization, \( M_b \) the magnitude of \( M \), \( M_0 = M(H_0, T_0) \) the constant mean value of magnetization, \( Q \) the rate of externally distributed magnetic field, \( \gamma \) the magnetic potential, \( K = -\frac{1}{2} \varepsilon (\mathbf{M} \cdot \frac{\partial \mathbf{M}}{\partial t}) \) the pyromagnetic co-efficient, \( \chi = (\mathbf{M} \cdot \frac{\partial \mathbf{M}}{\partial t}) \) the magnetic susceptibility, \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) the Laplacian operator, and the subscripts \( s \) and \( f \) represent the solid and fluid, respectively.

We assume that the magnetization is aligned with the magnetic field, but allow a dependence on the magnitude of magnetic field as well as the temperature in the form

$$\mathbf{M} = \mathbf{H} \mathbf{M}(H, T).$$  (7)

The magnetic equation of state is linearized about \( H_b \) and \( T_b \) to become

$$\mathbf{M} = M_b + \gamma (H - H_b) - K(T - T_b).$$  (8)

The basic state is assumed to be quiescent and is given by

$$\mathbf{V} = 0, \quad p = p_b(z), \quad p = p_b(z), \quad T = T_b(z), \quad \mathbf{H} = \mathbf{H}_b(z), \quad \mathbf{M} = \mathbf{M}_b(z).$$  (9)

Using Eq. (8), then Eqs. (2) and (3), respectively, yield

$$\frac{dp}{dz} = -\rho_0 [1-\chi(T_b-T_0)] g + \mu_0 M_b \frac{dH_b}{dz}.$$  (10)

$$\frac{d^2 t^2}{dz^2} = -\frac{Q}{k_1}.$$  (11)

Solving Eq. (10) subject to the boundary conditions \( T_b = T_0 \) at \( z = 0 \) and \( T_b = T_0 - \Delta T \) at \( z = d \), we obtain

$$T_b(z) = \frac{-Q z^2}{2k_1} + \frac{Q dz}{2k_1} - \beta z + T_0$$  (12)

where \( \beta = \Delta T/\Delta z \) is the temperature gradient.

Substituting Eq. (6) into Eq. (5a) and using Eq. (12), the basic state magnetic field intensity \( \mathbf{H}_b(z) \) and magnetization \( \mathbf{M}_b(z) \) are found to be (see Finlayson [6])

$$\mathbf{H}_b(z) = \left[ H_0 - \frac{K}{1+\chi} \left( \frac{Q z^2}{2k_1} - \frac{Q dz}{2k_1} + \beta z \right) \right] \hat{k}$$  (13)

$$\mathbf{M}_b(z) = \left[ M_0 + \frac{K}{1+\chi} \left( \frac{Q z^2}{2k_1} - \frac{Q dz}{2k_1} + \beta z \right) \right] \hat{k}$$  (14)

where \( M_0 + H_0 = H_c^c \).

Using Eqs. (12)–(14) in Eq. (10) and integrating, we obtain

$$\frac{dp_b(z)}{dz} = \rho_0 \frac{\partial \mathbf{g}}{\partial z} - \rho_0 \alpha z \left( \frac{Q z^2}{2k_1} - \frac{Q dz}{2k_1} + \beta z \right) \frac{\partial \mathbf{g}}{\partial z} \hat{r}$$  (15)

The pressure distribution is of no consequence here as we are eliminating the same. It may be noted that \( T_b(z), \mathbf{H}_b(z) \) and \( \mathbf{M}_b(z) \) are distributed parabolically with the porous layer height due to the presence of internal heat generation. However, when \( Q = 0 \) (i.e., in the absence of internal heat generation), the basic state temperature distribution is linear in \( z \). Thus the presence of internal heat generation plays a significant role on the stability of the system.

To study the linear stability of the above solution, the variables are perturbed in the form

$$\mathbf{V} = \mathbf{V}^e \mathbf{r} = p_b(z) + \mathbf{r}, \quad T = T_b(z) + \mathbf{r},$$

$$\mathbf{H} = \mathbf{H}_b(z) + \mathbf{H}^e, \quad \mathbf{M} = \mathbf{M}_b(z) + \mathbf{M}^e$$  (16)

where the primed quantities represent the perturbed variables. Substituting Eq. (16) into Eqs. (7) and (8) and using Eq. (6), we get (after dropping the primes)

$$H_x + M_x = (1 + \mu_0 H_0) H_x$$

$$H_y + M_y = (1 + \mu_0 H_0) H_y$$

$$H_z + M_z = (1 + \mu_0 H_0) H_z$$

$$H_x = M_x = (1 + \chi) H_x - KT$$

where \( (H_x, H_y, H_z) \) and \( (M_x, M_y, M_z) \) are the \( x, y, z \) components of the magnetic field and magnetization, respectively. In obtaining the above equations, it is assumed that \( k_0 = (1 + \chi) H_0 \) and \( k_0 Q = 2k_1 + \Delta T/\Delta T_0 \). Thus, the analysis is restricted to physical situations in which the magnetization induced by the variations in temperature gradient and internal heating is small compared to that induced by the external magnetic field.

Substituting Eq. (16) into momentum Eq. (2), linearizing, eliminating the pressure term by operating curl twice and using Eq. (17), the \( z \)-component of the resulting equation can be obtained as (after dropping the primes)

$$\left( \rho_0 C_v + \frac{\mu_0 H_0}{\chi} - \mu_0 \frac{\partial \mathbf{g}}{\partial z} \right) \mathbf{V}_w = \mu_0 K \left( \frac{Q z^2}{2k_1} + \frac{Q dz}{2k_1} - \beta \right) \mathbf{V}_w + \mu_0 \alpha x \mathbf{g} \mathbf{V}_c T$$

$$\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \right) \mathbf{V}_w$$  (18)

where \( w \) is the vertical component of perturbed seepage velocity and \( \mathbf{V}_c = \mathbf{g} \mathbf{V}_c / \mathbf{g} \) is the horizontal Laplacian operator.

The energy Eq. (3), after using Eq. (16) and linearizing, takes the form (after dropping the primes)

$$\left( \rho_0 C_v \frac{\partial T}{\partial T} - \mu_0 T_0 K \frac{\partial \mathbf{g}}{\partial T} \right) \mathbf{V}_w = k_1 \mathbf{V}_w + \left( \frac{\mu_0 T_0 K^2}{1 + \chi} \right) \left( \frac{Q z}{k_1} + \frac{Q dz}{2k_1} + \beta \right) \mathbf{V}_c T$$  (19)
where \((\rho_0 C_D) = \varepsilon \rho_0 C_{V_H} + \mu_0 \mu_0 H K + (1 - \varepsilon) (\rho_0 C_D)\), and \((\rho_0 C_D) = \varepsilon \rho_0 C_{V_H} + \mu_0 H K\)

Eq. (5a,b), after substituting Eq. (16) and using Eq. (17) may be written as (after dropping the primes)

\[
\left(1 + \frac{M_0}{H_0}\right) \nabla^2 \phi + (1 + \chi) \frac{\partial^2 \phi}{\partial t^2} - k_1 \frac{\partial T}{\partial z} = 0. \tag{20}
\]

Eqs. (18)-(20) are to be solved subject to appropriate boundary conditions on the perturbed variables. The lower boundary is considered to be rigid-ferromagnetic and insulated to temperature perturbations, while the upper boundary is free-ferromagnetic at which general thermal convection condition on perturbed temperature is invoked.

Accordingly, the boundary conditions are:

\[
w = \frac{\partial w}{\partial z} = 0, \quad \frac{\partial T}{\partial z} = 0, \quad (1 + \chi) \frac{\partial \phi}{\partial z} - a \phi = 0 \at z = 0 \tag{21a}
\]

\[
w = \frac{\partial^2 w}{\partial z^2} = 0, \quad k_1 \frac{\partial T}{\partial z} = h_1 T, \quad (1 + \chi) \frac{\partial \phi}{\partial z} + a \phi = 0 \at z = 1. \tag{21b}
\]

Performing the normal mode expansion of the dependent variables is assumed in the form

\[
w, T, \phi = [W(z), \Theta(z), \Phi(z)] \exp[i(\xi x + my) + \omega t] \tag{22}
\]

where \(\xi \) and \(\eta \) are wave numbers in the \(x\) and \(y\) directions, respectively, \(W(z), \Theta(z), \Phi(z)\) are, respectively, the amplitude of \(z\)-component of perturbation velocity, perturbation temperature, perturbation magnetization and \(\omega \) is the growth rate.

On substituting Eq. (22) into Eqs. (18)-(20), we get

\[
\left(\rho_0 \omega^2 \frac{H_1}{K} - \mu_0 (D^2 - a^2) \right) \left(\frac{D^2}{D^2 - a^2}\right) M = -a^2 \mu_0 K \left(\frac{Q_0}{k_1^2} - \frac{4}{4 - 4 \beta} \right) \phi - \phi_1 \frac{D^2}{D^2 - a^2} \phi \frac{D^2}{D^2 - a^2} \phi + a^2 \mu_0 \kappa^2 \left(\frac{Q_0}{k_1^2} - \frac{4}{4 - 4 \beta} \right) \Theta \tag{23}
\]

\[
\left[\rho_0 (\omega \phi - \frac{H_1}{K} - \mu_0 \Theta) - \mu_0 T_0 k_0 \alpha D = k_1 (D^2 - a^2) \Theta + \left[\rho_0 (\omega \phi - \frac{H_1}{K} - \mu_0 \Theta) - \mu_0 T_0 k_0 \alpha D \right] \tag{24}
\]

\[
(1 + \chi) D^2 \phi - \left(1 + \frac{M_0}{H_0}\right) a^2 \phi - K D \Theta = 0 \tag{25}
\]

where \(D = \frac{D}{Dx} \) is the differential operator and \(a = \sqrt{\gamma^2 + m^2} \) is the overall horizontal wave number. Noting that the principle of exchange of stability is valid (see [30]) and non-dimensionalizing the variables by setting

\[
(x^*, y^*, z^*) = \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right), \quad W^* = \frac{a}{a} W, \quad a^* = a d, \quad D^* = a D, \quad \Theta = \frac{\kappa}{\rho \nu \alpha}, \quad \Phi^* = \left(\frac{1 + \chi}{\kappa} \right) \Phi, \quad \omega^* = \frac{\omega}{d}, \alpha = \frac{\alpha}{d} \tag{26}
\]

where \(A = (\rho_0 C_D) / (\rho_0 C_D)\) is the ratio of heat capacities, we obtain (after dropping the asterisks)

\[
\left[\frac{d}{d^2} (D^2 - a^2)\right] - M_1 (1 - 2z) - M_2 A) W = a^2 R_m (N_i (1 - 2z) - 1) (D \Theta + \alpha^2 R_i \Theta \tag{27}
\]

\[
(D^2 - a^2) \Theta = N_i (1 - 2z) \tag{28}
\]

\[
(D^2 - a^2) \Theta = N_i (1 - 2z) \tag{29}
\]

where \(R_t = \gamma x, \beta x d^2 / K \) is the thermal Rayleigh number and it is the ratio of buoyant to viscous forces, \(R_n = R_s M_1 = \mu_0 K^2 - d \) is the magnetic Rayleigh number and it represents the ratio of magnetic to viscous forces, \(M_1 = \mu_0 K^2 / (1 + \chi) \) is the magnetic number and it represents the ratio of magnetic to gravitational forces, \(N_i = Q_0 / Q_2 \) is the dimensionless heat source strength, \(M_2 = \mu_0 T_0 K^2 / (1 + \chi) \) is the magnetic parameter, \(M_3 = (1 + M_0 / H_0) / (1 + \chi) \) represents the measure of nonlinearity

of the magnetization, \(A = \mu_i / \mu_0 \) is the ratio of viscosities and \(D = 1 / d^2 \) is the Darcy number. The typical value of \(M_2 \) for magnetic fluids with different carrier liquids turns out to be of the order of \(10^{-6} \) and hence its effect is neglected as compared to unity.

The boundary conditions for perturbed non-dimensional variables take the following form:

\[
W = D W = D \Theta = (1 + \chi) D \Phi - a \Phi = 0 \at z = 0 \tag{30a}
\]

\[
W = D^2 W = D \Theta + \alpha D \Phi + a \Phi = 0 \at z = 1 \tag{30b}
\]

where \(B_i = (\alpha H_d / k_1) \) is the Biot number. The case \(B_i = 0 \) and \(B_i \rightarrow \infty \) correspond respectively to constant heat flux and isothermal conditions at the upper boundary.

3. Method of Solution

Eqs. (27)-(29) together with boundary conditions given by Eqs. (30a,b) constitute an eigenvalue problem with thermal Rayleigh number \(R_t \) being an eigenvalue. Accordingly, \(W, \Theta \) and \(\Phi \) are written as

\[
W(z) = \sum_{i=1}^{n} A_i W_i(z), \quad \Theta(z) = \sum_{i=1}^{n} B_i \Theta_i(z) \quad \Phi(z) = \sum_{i=1}^{n} C \Phi_i(z) \tag{31}
\]

Where \(A_i, B_i, C_i \) are unknown constants to be determined. The basis functions \(W_i(z), \Theta_i(z), \Phi_i(z) \) are generally chosen such that they satisfy the corresponding boundary conditions. Substituting Eq. (31) into Eqs. (27)-(29), multiplying the resulting momentum Eq. (27) by \(W_i(z) \), the energy Eq. (28) by \(\Theta_i(z) \) and the magnetic potential Eq. (29) by \(\Phi_i(z) \); performing the integration by parts with respect to \(z \) between \(z = 0 \) and \(z = 1 \) and using the boundary conditions (30a,b), we obtain the following system of linear homogeneous algebraic equations:

\[
C_{ij} A_i + D_{ij} B_i + E_{ij} C_i = 0 \tag{32}
\]

\[
F_{ij} A_i + G_{ij} B_i = 0 \tag{33}
\]

\[
H_{ij} B_i + I_{ij} C_i = 0 \tag{34}
\]

The coefficients \(C_{ij} \sim I_{ij} \) involve the inner products of the basis functions and are given by

\[
C_{ij} = \int \text{D}^2 W_i \cdot \text{D} \cdot \text{D} W_j + \int \text{D}^2 W_i \cdot \text{D} \cdot \text{D} W_j + \int \text{D}^2 W_i \cdot \text{D} \cdot \text{D} W_j \tag{35}
\]

\[
D_{ij} = \int \text{D} \cdot \text{D} W_i \cdot \text{D} \Theta_j + \int \text{D} \cdot \text{D} W_i \cdot \text{D} \Theta_j + \int \text{D} \cdot \text{D} W_i \cdot \text{D} \Theta_j \tag{36}
\]

\[
E_{ij} = \int \text{D} \cdot \text{D} \Theta_i \cdot \text{D} \Phi_j + \int \text{D} \cdot \text{D} \Theta_i \cdot \text{D} \Phi_j + \int \text{D} \cdot \text{D} \Theta_i \cdot \text{D} \Phi_j \tag{37}
\]

where the inner product is defined as \(\langle \ldots \rangle = \int_0^1 (\ldots) dz \)

The above set of homogeneous algebraic equations can have a non-trivial solution if and only if

\[
C_{ij} = D_{ij} = E_{ij} \tag{35}
\]

The eigenvalue can be extracted from the above characteristic equation. For this, we select the following trial functions

\[
W_1 = (z^2 - 2 \lambda^2 / 2 \lambda^2 (2 / 2)) T_{\lambda^2 - 1}, \quad \Theta_1 = (z - 2 \lambda^2 / 2 \lambda^2 (2 / 2)) T_{\lambda^2 - 1}, \quad \Phi_1 = (z - 1) T_{\lambda^2 - 1} \tag{37}
\]

where \(T_{\lambda^2} \) 's are the modified Chebyshev polynomials. The above trial functions satisfy all the boundary conditions except the
condition \((1 + \chi)D\Phi - \alpha\Phi = 0\) at \(z = 0\) and \(D\Theta + B\Theta = 0 = (1 + \chi)D\Phi + \alpha\Phi\) at \(z = 1\), but the residuals from these conditions are included as residuals from the differential equation. Eq. (36) leads to a relation involving the physical parameters \(R_c, N_s, A, Da^{-1}, M_1, M_2, B, \chi\) and \(a\) in the form
\[
f(R_c, N_s, Da^{-1}, A, Bi, M_1, M_2, \chi, a) = 0.
\]

The critical value of \(R_c\) (i.e., \(R_c^\ast\)) is determined numerically with respect to \(a\) for different values of \(N_s, A, Da^{-1}, M_1, M_2\) and \(\chi\).

4. Numerical Results and Discussion

The linear stability analysis has been carried out to investigate penetrative ferroconvection via internal heating in a horizontal ferrofluid saturated Brinkman porous layer. The lower boundary is rigid-insulating to temperature perturbations while a general thermal convection boundary condition on perturbed temperature is invoked on the upper free boundary. The critical Rayleigh number \(R_c\) and the corresponding critical wave number \(a\) are obtained numerically using the Galerkin technique for various values of physical parameters \(N_s, A, Da^{-1}, M_1, M_2, \chi, a\) and \(Bi\). The convergence is achieved by considering six terms (i.e., \(i = j = 6\)) in the Galerkin expansion.

To validate the numerical procedure used in the present study, the critical Rayleigh number \(R_c\) and the corresponding critical wave number \(a\) obtained under the limiting case of \(M_1 = 0, Da^{-1} = 0\) and \(N_s = 0\) (ordinary viscous fluid layer) for different values of Biot number \(Bi\) are compared with those of Sparrow et al. [39] in Table 1. We note that the agreement is good and thus verify the accuracy of the numerical procedure employed.

The presence of internal heating makes the basic temperature, magnetic field and magnetization distributions to deviate from linear to parabolic with respect to porous layer height which in turn have significant influence on the stability of the system. To assess the impact of internal heat source strength \(N_s\) on the criterion for the onset of ferroconvection, the distributions of dimensionless basic temperature, \(\tilde{T}_b(z)\), magnetic field intensity, \(H_L(z)\) and magnetization, \(M_L(z)\) are exhibited graphically in Fig. 2 for different values of \(N_s\). From the figure it is observed that increase in the internal heat source strength amounts to large deviations in these distributions which in turn enhance the disturbances in the porous layer and thus reinforce instability on the system.

The critical Rayleigh number \(R_c\) and the corresponding critical wave number \(a\) are presented graphically in Figs. 3(a) and (b) respectively show the variation of \(R_c\) and the corresponding \(a\) as a function of inverse of Darcy number \(Da^{-1}\) for different values of heat transfer coefficient \(Bi\) with fixed values of \(N_s = 2, A = 2, M_1 = 2, M_2 = 1\) and \(\chi = 0\). It is seen that the \(R_c\) increases with increasing \(Da^{-1}\) and hence its effect is to delay the onset of ferroconvection. For a fixed thickness of the porous layer, increase in the value of \(Da^{-1}\) leads to decrease in the permeability of the porous medium which in turn decelerate the fluid motion in a ferrofluid saturated porous medium. Besides, it is observed that an increase in the value of heat transfer coefficient \(Bi\) is to increase the critical Rayleigh number and thus its effect is to delay the onset of ferroconvection. This may be attributed to the fact that with increasing \(Bi\), the thermal disturbances can easily dissipate into the ambient surrounding due to a better convective heat transfer coefficient at the top surface and hence higher heating is required to make the system unstable. Fig. 3(b) represents the corresponding critical wave number \(a\) and it indicates that increase in the value of \(Bi\) and \(Da^{-1}\) is to increase \(a\) and thus their effect is to reduce the size of convection cells.

The model that we have considered to describe the flow in a porous medium rests on an effective viscosity different from fluid viscosity. In the present study, the ratio of these two viscosities is taken as a separate parameter, denoted by \(A\), and found that it has a profound influence on the onset of ferroconvection in a ferrofluid saturated porous medium. Fig. 4(a) and (b), respectively represent the variation of \(R_c\) and the corresponding \(a\) as a function of \(Da^{-1}\) for various values of \(A\) when \(N_s = 2, Bi = 2, M_1 = 2, M_2 = 1\) and \(\chi = 0\). From Fig. 4(a), it is observed that an increase in the value of viscosity ratio \(A\) is to delay the onset of ferroconvection. This is because; increase in the value of \(A\) is related to increase in viscous effect which has the tendency to retard the fluid flow and hence higher heating is required for the onset of ferroconvection. In other words, higher value of \(A\) is more effective in suppression of ferroconvection in a ferrofluid saturated porous medium. To the contrary, increase in the value of \(A\) is to decrease \(a\) and thus its effect is to increase the size of convection cells (see Fig. 4(b)).

The effect of magnetic number \(M_1\) on the onset of ferroconvection is made clear in Fig. 5(a) by presenting the critical Rayleigh number \(R_c\) as a function of \(Da^{-1}\) for different values of \(M_1\) when \(N_s = 2, Bi = 2, A = 2, M_2 = 1\) and \(\chi = 0\). It is seen that an increase in the value of magnetic number \(M_1\) is destabilizing. Physically,
increase in $M_1$ leads to either increase in destabilizing magnetic force or decrease in stabilizing viscous force on the system and hence it has a destabilizing effect on the system. A closer inspection of the figure further reveals that the magnetic force is to reinforce together with buoyancy force and to hasten the onset of ferroconvection when compared to their effect in isolation. Fig. 5(b) illustrates that increase in $M_1$ is to increase $a_\xi$ and thus leads to reduce the dimension of the convection cells.

The measure of non-linearity of fluid magnetization, denoted through the parameter $M_3$, on the onset of ferroconvection in a ferrofluid saturated porous layer is depicted in Fig. 6(a). The curves of $R_{tc}$ versus $Da^{-1}$ shown in Fig. 6(a) for different values of $M_3$ when $N_f=2, A=2, M_1=2, M_3=1$ and $\chi=0$. As $M_3$ increases, the curve shifts monotonously between those for $M_3=0$ and $\infty$. The destabilization due to increase in the nonlinearity of the fluid magnetization is not so significant at lower values of $Da^{-1}$ but it is more apparent with increasing $Da^{-1}$. This may be attributed to the fact that the application of magnetic field makes the ferrofluid to acquire larger magnetization which in turn interacts with the imposed magnetic field and releases more energy to drive the flow faster. Hence, the system becomes unstable with a smaller temperature gradient as the value of $M_3$ increases. Moreover, the results reduces to the classical ordinary viscous fluid case ($R_m=0$) in the absence of internal heating ($N_i=0$) and buoyancy force ($R_t=0$) as $M_3 \to \infty$. This fact is evident from the critical stability parameters tabulated in Table 2 for different values of $Da^{-1}$ and for two values of $\chi$. However, the above observed result is found to be not true in the presence of internal heating. From Fig. 6(b), we note that increase in the value of $M_3$ is to increase the critical wave number and thus to decrease the size of convection cells.

To explore the effect of dimensionless internal heat source strength $N_i$ on the criterion for the onset of ferroconvection, the variation of $R_{tc}$ and $a_\xi$ is exhibited as a function of $Da^{-1}$ in Fig. 7(a) and (b), respectively for different values of dimensionless internal heat source strength $N_i$. Fig. 7(a) clearly indicates that $R_{tc}$ decreases monotonically with $N_i$ indicating the influence of increasing internal heating is to decrease the value of $R_{tc}$ and thus destabilize the system. This is because; increasing $N_i$ amounts to increase in energy supply to the system. Eventually, this leads to large deviations in the basic state temperature.
distribution (see Fig. 2) of the parabolic type which in turn enhances the thermal disturbances in the ferrofluid saturated porous layer to hasten the onset of ferroconvection. Fig. 7(b) reveals that the critical wave number \( a_c \) increases monotonically with an increase in the value of \( N_s \) and thus its effect is to reduce the dimension of convection cells.

The complementary effects of both buoyancy and magnetic forces are made clear in Fig. 8 by displaying the locus of the critical thermal Rayleigh number \( R_{tc} \) and magnetic Rayleigh number \( R_{mc} \) for different values of \( M_3 \) when \( N_s = 2, Bi = 2, A = 2, M_3 = 1 \) and \( \chi = 0 \). We note that \( R_{tc} \) is inversely proportional to \( R_{mc} \) due to the destabilizing magnetic force. As \( M_3 \to \infty \) the data does not fit the empirical equation \( R_{tc}/R_{mc} = 1 \), where \( R_{mc0} \) is the critical magnetic Rayleigh number in the non-gravitational case \( (R_{mc} = 0) \) and \( R_{mc0} \) is the critical magnetic Rayleigh number in the non-magnetic case \( (R_{mc} = 0) \), but otherwise found to be true in the absence of internal heating [27]. Moreover, the system is found to be more stable with increasing \( \chi \) compared to \( \chi = 0 \).

A closer inspection of the figure further depicts that the deviation in the \( R_{tc} \) values for different magnetic boundary conditions is more pronounced with increasing coupling parameter. To know the impact of different temperature and magnetic boundary conditions on the stability characteristics of the system, the results of the present study are compared with those of Nanjundappa et al. [35] in Fig. 9. The variation of critical Rayleigh number as a function of \( N_s \) for two values of \( w \) when \( Bi = 2, Da^{-1}/C_0 = 100, L = 2, M_3 = 1 \) and \( M_1 = 2 \) is presented in the figure. Compared to the present set of temperature and magnetic boundary conditions, it is observed that the system is more stable when both boundaries are ferromagnetic and the lower boundary is conducting as considered in [35]. Moreover, it is seen that the curves of critical Rayleigh number for different values of \( w \) and for different temperature and magnetic boundary conditions coalesce with increasing internal heat source strength.

5. Conclusions

The onset of penetrative ferroconvection via internal heating in a ferrofluid saturated Brinkman porous layer is investigated.
The lower boundary is considered to be rigid – insulating to temperature perturbations while the upper boundary is free and subject to a general thermal condition on the perturbed temperature. The resulting eigenvalue problem is solved numerically by employing the Galerkin technique. The effect of internal heating is to alter the basic temperature distribution from linear to parabolic with respect to porous layer height and found that it has a profound effect on the stability characteristics of the system.

The following conclusions can be drawn from the present study:

(i) The effect of increase in the internal heat source strength \( N_s \) is to lower the critical thermal Rayleigh number \( R_{tc} \) and hence to hasten the onset of ferroconvection in a ferrofluid saturated porous layer. The critical stability parameters do not fit the empirical equation \( R_{tc}/R_{tc0} + R_{mc}/R_{mc0} = 1 \) as \( M_3 \to \infty \) in the presence of internal heating but otherwise found to be true in its absence.

(ii) The system becomes more unstable with an increase in the value of magnetic number \( M_1 \) and nonlinearity of fluid magnetization parameter \( M_3 \).

![Fig. 7](image1.png)  
**Fig. 7.** Variation of (a) \( R_{tc} \) and (b) \( a_c \) as a function of \( Da^{-1} \) for different values of \( N_s \) when \( Bi=2, A=2, M_1=2, M_3=1 \) and \( \chi = 0 \).

![Fig. 8](image2.png)  
**Fig. 8.** Variation of \( R_{mc} \) as a function of \( Da^{-1} \) with different values of \( M_3 \) and \( \chi \) for \( Bi=2, Da^{-1}=100, A=1, M_1=2 \) and \( N_s=2 \).

![Fig. 9](image3.png)  
**Fig. 9.** Variation of \( R_{tc} \) as a function of \( N_s \) with different values of \( \chi \) for \( Bi=2, Da^{-1}=100, A=1, M_1=2 \) and \( N_s=0 \).

**Table 2**  
Comparison of \( R_{tc}, R_{mc} \) and \( a_c \) for different values of \( Da^{-1}, M_1 \) and \( \chi \) when \( Bi=2, A=1 \) and \( N_s=0 \).

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>( Da^{-1} )</th>
<th>( R_{mc}, R_{mc0} )</th>
<th>( M_1=1 )</th>
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(iii) The critical thermal Rayleigh number \( R_{tc} \) increases with an increase in the value of Biot number \( Bi \), ratio of viscosities \( \lambda \), inverse Darcy number \( Da/C_0 \), as well as magnetic susceptibility \( w \) and thus their effect is to delay the onset of ferroconvection.

(iv) The effect of increase in \( Bi \), \( Da/C_0 \), \( M_1 \) and \( Ns \), and decrease in \( L \) and \( M_3 \) is to increase the critical wave number \( ac \) and hence their effect is to decrease the dimension of convection cells.

(v) It is possible to either augment or suppress ferroconvection in a porous medium by tuning the physical parameters of the system.

Acknowledgments

The authors CEN and HNP express their heartfelt thanks to the Principals of their respective colleges for their encouragement. We thank the reviewers for their useful comments which have led to the improvement of the paper.

References