Uniqueness theorems on meromorphic functions sharing one value

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Received 24 May 2007; received in revised form 13 November 2007; accepted 17 November 2007

Abstract

In this paper, we study with a weighted sharing method the uniqueness problem of \( [f^n(z)]^k \) and \( [g^n(z)]^k \) sharing one value and obtain some results which extend the theorems given by M. Fang, S. Bhoosnurmath and S. Dyavanal et al.

Keywords: Meromorphic function; Entire function; Weighted sharing; Derivative; Uniqueness

1. Introduction and main results

In this paper, a meromorphic function means meromorphic in the open complex plane. We shall adopt the standard notations in Nevanlinna’s value distribution theory of meromorphic functions such as the characteristic function \( T(r, f) \), the counting function of the poles \( N(r, f) \), and the proximity function \( m(r, f) \) (see [1,2]).

Let \( f \) and \( g \) be two nonconstant meromorphic functions, \( a \in \mathbb{C} \cup \{\infty\} \). We say that \( f \) and \( g \) share the value \( a \) IM (Ignoring Multiplicities) if \( f - a \) and \( g - a \) have the same zeros. Moreover, if \( f - a \) and \( g - a \) have the same zeros with the same multiplicities, we say that they share the value \( a \) CM (Counting Multiplicities). When \( a = \infty \) the zeros of \( f - a \) means the poles of \( f \). A meromorphic function \( a (\not\equiv \infty) \) is called a small function with respect to \( f \) provided that \( T(r, a) = S(r, f) \). Note that the set of all small functions of \( f \) is a field. Let \( b \) be a small function with respect to \( f \) and \( g \). We say that \( f \) and \( g \) share \( b \) IM(CM) provided that \( f - b \) and \( g - b \) have the same zeros ignoring(counting) multiplicities. Let \( p \) be a positive integer. We use \( N_p(r, \frac{1}{f-a}) \) to denote the counting function of the zeros of \( f - a \) whose multiplicities are not greater than \( p \), \( N_p(r, \frac{1}{f-a}) \) to denote the counting function of the zeros of \( f - a \) whose multiplicities are not less than \( p \). And \( \overline{N}_p(r, \frac{1}{f-a}) \), \( \overline{N}_p(r, \frac{1}{f-a}) \) are their reduced functions, respectively. We also use \( N_p(r, \frac{1}{f-a}) \) to denote the counting function of the zeros of \( f - a \) where a zero with multiplicity \( m \) is counted \( m \) times if \( m \leq p \) and \( p \) times if \( m > p \). Clearly, \( N_1(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) \). Define

\[
\delta_p(a, f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, \frac{1}{f-a})}{T(r, f)}.
\]

Obviously, \( 1 \geq \Theta(a, f) \geq \delta_p(a, f) \geq \delta(a, f) \geq 0 \).

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Hayman [3] and Clunie [4] proved the following result:

**Theorem A.** Let \( f(z) \) be a transcendental entire function, \( n \geq 1 \) a positive integer, then \( f^n f' = 1 \) has infinitely many solutions.

Fang and Hua [5] and Yang and Hua [6] obtained a unicity theorem corresponding to the above result.

**Theorem B.** Let \( f(z) \) and \( g(z) \) be two nonconstant entire functions, \( n \geq 6 \) a positive integer. If \( f^n f' \) and \( g^6 g' \) share \( 1 \) CM, then either \( f(z) = c_1 e^{cz} \), \( g(z) = c_2 e^{-cz} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \((c_1c_2)^n + c^2 = -1\) or \( f(z) = tg(z) \) for a constant \( t \) such that \( t^{n+1} = 1 \).

Hennekemper [7] and Chen [8] extended Theorem A as follows:

**Theorem C.** Let \( f(z) \) be a transcendental entire function, \( n, k \) be two positive integers satisfying \( n \geq k + 1 \), then \([f^n(z)]^{(k)} = 1 \) has infinitely many solutions.

Recently, Bhoosnurmath and Dyavanal [9] obtained a unicity theorem corresponding to Theorem C.

**Theorem D.** Let \( f(z) \) and \( g(z) \) be two nonconstant meromorphic functions, and let \( n, k \) be two positive integers with \( n > 3k + 8 \). If \([f^n(z)]^{(k)} \) and \([g^n(z)]^{(k)} \) share \( 1 \) CM, then either \( f(z) = c_1 e^{cz} \), \( g(z) = c_2 e^{-cz} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \((-1)^k(c_1c_2)^n(nc)^{2k} = 1\) or \( f(z) = tg(z) \) for a constant \( t \) such that \( t^n = 1 \).

Naturally, one may ask the following question: *Is it really possible to relax in any way the nature of sharing 1 in the above results?*

The purpose of this paper is to discuss this problem. To do this, we now introduce the notation of weighted sharing (see [10]).

**Definition 1.** Let \( k \) be a non-negative integer or infinity. For \( a \in C \cup \{\infty\} \) we denote by \( E_k(a, f) \) the set of all \( a \)-points of \( f \) where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \).

**Definition 2.** Let \( k \) be a non-negative integer or infinity. If for \( a \in C \cup \{\infty\} \) such that \( E_k(a, f) = E_k(a, g) \), then we say that \( f \) and \( g \) share the value \( a \) with weight \( k \).

We write \( f, g \) sharing \((a, k)\) to mean that \( f, g \) share the value \( a \) with weight \( k \). Clearly, if \( f, g \) share \((a, k)\) then \( f, g \) share \((a, p)\) for all integer \( p, 0 \leq p \leq k \). Also we note that \( f, g \) share a value \( a \) CM if and only if \( f, g \) share \((a, \infty)\).

In this paper, using the idea of weighted sharing of values introduced by I. Lahiri, we will study the problem that \([f^n(z)]^{(k)} \) and \([g^n(z)]^{(k)} \) sharing one value with the weighted sharing method and obtain the following theorems, which improve and extend the above theorems.

**Theorem 1.** Let \( f(z) \) and \( g(z) \) be two nonconstant transcendental meromorphic functions, and let \( n(\geq 1), k(\geq 1), \ell(\geq 0) \) be three integers. Suppose that \([f^n(z)]^{(k)} \) and \([g^n(z)]^{(k)} \) share \((1, \ell)\), if \( l \geq 2 \) and \( n > 3k + 8 \) or if \( l = 1 \) and \( n > 5k + 11 \) or \( l = 0 \) and \( n > 9k + 14 \), then either \( f(z) = c_1 e^{cz} \), \( g(z) = c_2 e^{-cz} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \((-1)^k(c_1c_2)^n(nc)^{2k} = 1\) or \( f(z) = tg(z) \) for a constant \( t \) such that \( t^n = 1 \).

**Theorem 2.** Let \( f(z) \) and \( g(z) \) be two nonconstant transcendental entire functions, and let \( n(\geq 1), k(\geq 1), \ell(\geq 0) \) be three integers. Suppose that \([f^n(z)]^{(k)} \) and \([g^n(z)]^{(k)} \) share \((1, \ell)\), if \( l \geq 2 \) and \( n > k + 3 \) or if \( l = 1 \) and \( n > 3k + 6 \) or \( l = 0 \) and \( n > 5k + 7 \), then either \( f(z) = c_1 e^{cz} \), \( g(z) = c_2 e^{-cz} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \((-1)^k(c_1c_2)^n(nc)^{2k} = 1\) or \( f(z) = tg(z) \) for a constant \( t \) such that \( t^n = 1 \).

2. Some lemmas

For the proof of our results, we need the following lemmas.

**Lemma 1** ([11,12]). Let \( f \) be a nonconstant meromorphic function and \( k \) be a positive integer, then

\[
N_p \left( r, \frac{1}{f^{(k)}} \right) \leq N_{p+k} \left( r, \frac{1}{f} \right) + k\overline{N}(r, f) + S(r, f) \leq (p + k)\overline{N} \left( r, \frac{1}{f} \right) + k\overline{N}(r, f) + S(r, f).
\]

This lemma can be obtained immediately from the proof of Lemma 2.3 in [11] which is the case \( p = 2 \).
Lemma 2 ([6]). Let \( f \) be a nonconstant meromorphic function and \( n \) be a positive integer. Also let \( P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f \), where \( a_i \) are meromorphic functions such that \( a_n \neq 0 \), \( T(r, a_i) = S(r, f)(i = 1, 2, \cdots, n) \). Then
\[
T(r, P(f)) = n T(r, f) + S(r, f).
\]

Lemma 3 ([11]). Let \( f(z) \) be a nonconstant meromorphic function and \( k \) a positive integer, and let \( c \) be a nonzero finite complex number. Then
\[
T(r, f) \leq \overline{N}(r, f) + N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f^{(k)} - c} \right) - N \left( r, \frac{1}{f^{(k+1)}} \right) + S(r, f)
\]
\[
\leq \overline{N}(r, f) + N_{k+1} \left( r, \frac{1}{f} \right) + \overline{N} \left( r, \frac{1}{f^{(k)} - c} \right) - N_0 \left( r, \frac{1}{f^{(k+1)}} \right) + S(r, f),
\]
where \( N_0(r, \frac{1}{f^{(k+1)}}) \) is the counting function which only counts those points such that \( f^{(k+1)} = 0 \) but \( f^{(k)}(z) \neq 0 \).

Lemma 4 ([13]). Let \( f(z) \) be a nonconstant entire function, and \( k \geq 2 \) be a positive integer. If \( f(z) f^{(k)}(z) \neq 0 \), then \( f = e^{az+b} \), where \( a(\neq 0) \) and \( b \) are constants.

Lemma 5. Let \( F \) and \( G \) be two nonconstant meromorphic functions. If \( F \) and \( G \) share \( 1 \) IM, then
\[
\overline{N}_L \left( r, \frac{1}{F - 1} \right) \leq \overline{N} \left( r, \frac{1}{F} \right) + \overline{N}(r, F) + S(r, F).
\]

Proof. Note that \( F \) and \( G \) share \( 1 \) IM, we have
\[
\overline{N}_L \left( r, \frac{1}{F - 1} \right) \leq \overline{N}_L \left( r, \frac{1}{F - 1} \right) \leq \bar{N} \left( r, \frac{1}{F - 1} \right) - \overline{N} \left( r, \frac{1}{F - 1} \right), \tag{2.1}
\]
\[
\bar{N} \left( r, \frac{1}{F - 1} \right) - \bar{N} \left( r, \frac{1}{F - 1} \right) + N \left( r, \frac{1}{F} \right) - \overline{N} \left( r, \frac{1}{F} \right) \leq N \left( r, \frac{1}{F} \right) \leq \overline{N}(r, F) + S(r, F), \tag{2.2}
\]
which is equivalent to
\[
\bar{N} \left( r, \frac{1}{F - 1} \right) - \bar{N} \left( r, \frac{1}{F - 1} \right) \leq \overline{N} \left( r, \frac{1}{F} \right) + \overline{N}(r, F) + S(r, F). \tag{2.3}
\]
Combining (2.1) and (2.3), we obtain
\[
\overline{N}_L \left( r, \frac{1}{F - 1} \right) \leq \overline{N} \left( r, \frac{1}{F} \right) + \overline{N}(r, F) + S(r, F).
\]

Obviously, the above inequality also holds with respect to \( G \). \( \square \)

3. Proof of Theorem 1

Set \( F(z) = f^n(z) \) and \( G(z) = g^n(z) \), then we have
\[
\Theta(0, F) = 1 - \lim_{r \to \infty} \sup \frac{\overline{N} \left( r, \frac{1}{F} \right)}{T(r, F)} = 1 - \lim_{r \to \infty} \sup \frac{\overline{N} \left( r, \frac{1}{F} \right)}{n T(r, f)} \geq \frac{n - 1}{n}. \tag{3.1}
\]
Similarly,
\[ \Theta(0, G) \geq \frac{n - 1}{n}. \] (3.2)
\[ \Theta(\infty, F) = 1 - \lim_{r \to \infty} \sup \frac{\overline{N}(r, F)}{T(r, F)} = 1 - \lim_{r \to \infty} \sup \frac{\overline{N}(r, f)}{nT(r, f)} \geq \frac{n - 1}{n}. \] (3.3)

Similarly,
\[ \Theta(\infty, G) \geq \frac{n - 1}{n}. \] (3.4)

Next, we have
\[ \delta_{k+1}(0, F) = 1 - \lim_{r \to \infty} \sup \frac{N_{k+1} \left( r, \frac{1}{F} \right)}{T(r, F)} \geq 1 - \lim_{r \to \infty} \sup \frac{(k + 1)\overline{N} \left( r, \frac{1}{F} \right)}{T(r, F)} \geq 1 - \lim_{r \to \infty} \sup \frac{(k + 1)\overline{N} \left( r, \frac{1}{F} \right)}{nT(r, f)} \geq 1 - \frac{k + 1}{n} = \frac{n - (k + 1)}{n}. \] (3.5)

Similarly,
\[ \delta_{k+1}(0, G) \geq \frac{n - (k + 1)}{n}. \] (3.6)

Let
\[ \varphi(z) = \left( F^{(k+2)} - 2 F^{(k+1)} F^{(k)} - 2 G^{(k+1)} G^{(k)} \right). \] (3.7)

Suppose that \( \varphi(z) \neq 0. \)
If \( z_0 \) is a common simple \( 1 \)-point of \( F^{(k)}(z) \) and \( G^{(k)}(z) \), substituting their Taylor series at \( z_0 \) into (3.7), we can get \( \varphi(z_0) = 0. \) Thus we have
\[ N^{(1)}_E \left( r, \frac{1}{F^{(k+1)}} \right) = N^{(1)}_E \left( r, \frac{1}{G^{(k+1)}} \right) \leq \overline{N} \left( r, \frac{1}{F} \right) \leq T(r, \varphi) + O(1) \leq N(r, \varphi) + S(r, F) + S(r, G), \] (3.8)
where \( N^{(1)}_E \left( r, \frac{1}{F^{(k+1)}} \right) \) denotes the counting function of common \( 1 \)-points of \( F^{(k)} \) and \( G^{(k)} \).

According to our assumption, \( \varphi(z) \) has simple poles only at zeros of \( F^{(k+1)}, F^{(k)} - 1 \) and \( G^{(k+1)}, G^{(k)} - 1 \) as well as poles of \( F \) and \( G \).

From Lemma 3, we have
\[ T(r, F) + T(r, G) \leq N_{k+1} \left( r, \frac{1}{F} \right) + N_{k+1} \left( r, \frac{1}{G} \right) + \overline{N} \left( r, \frac{1}{F^{(k+1)}} \right) + \overline{N} \left( r, \frac{1}{G^{(k+1)}} \right) + \overline{N}(r, F) + \overline{N}(r, G). \] (3.9)

Obviously,
\[ N \left( r, \frac{1}{F^{(k+1)}} \right) \leq T(r, F^{(k)}) + O(1) \leq T(r, F) + k \overline{N}(r, F) + S(r, F). \] (3.10)

If \( l \geq 2 \), we have
\[ N(r, \varphi) \leq \overline{N}(r, F) + \overline{N}(r, G) + \overline{N} \left( r, \frac{1}{F} \right) + \overline{N} \left( r, \frac{1}{G} \right). \]
\[ + \mathcal{N}_{l+1} \left( r, \frac{1}{F(k) - 1} \right) + N_0 \left( r, \frac{1}{F(k+1)} \right) + N_0 \left( r, \frac{1}{G(k+1)} \right), \]  

(3.11)

and

\[ \mathcal{N}_{l+1} \left( r, \frac{1}{F(k) - 1} \right) + \mathcal{N} \left( r, \frac{1}{F(k) - 1} \right) + \mathcal{N} \left( r, \frac{1}{G(k) - 1} \right) \leq N_{l+1} \left( r, \frac{1}{G(k) - 1} \right) + N \left( r, \frac{1}{F(k) - 1} \right). \]  

(3.12)

From (3.8)–(3.12) we deduce that

\[ T(r, G) \leq (k + 2)\mathcal{N}(r, F) + 2\mathcal{N}(r, G) + \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{G} \right) + N_{k+1} \left( r, \frac{1}{F} \right) + N_{k+1} \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G). \]

Without loss of generality, we suppose that there exists a set \( I \) with infinite linear measure such that

\[ T(r, F) \leq T(r, G) \]

for \( r \in I \). Hence

\[ T(r, G) \leq [(k + 2)(1 - \Theta(\infty, F)) + 2(1 - \Theta(\infty, G)) + (1 - \Theta(0, F)) + (1 - \Theta(0, G)) + (1 - \delta_{k+1}(0, F)) + (1 - \delta_{k+1}(0, G)) + \varepsilon]T(r, G) + S(r, G), \]

for \( r \in I \) and \( 0 < \varepsilon < \Delta_1 - (k + 7) \), that is

\[ [\Delta_1 - (k + 7) - \varepsilon]T(r, G) \leq S(r, G), \]

i.e.,

\[ \Delta_1 \leq k + 7, \]  

(3.13)

where

\[ \Delta_1 = (k + 2)\Theta(\infty, F) + 2\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G) \]

\[ \geq (k + 2 + 2) \frac{n - 1}{n} + 2 \cdot \frac{n - 1}{n} + 2 \cdot \frac{n - (k + 1)}{n} \]

\[ = k + 8 - \frac{3k + 8}{n}. \]

Note that (3.13), we have \( n \leq 3k + 8 \), which contradicts our hypothesis \( n > 3k + 8 \).

If \( l = 1 \), then

\[ N(r, \varphi) \leq \mathcal{N}(r, F) + \mathcal{N}(r, G) + \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{G} \right) + N(2) \left( r, \frac{1}{F(k) - 1} \right) + N_0 \left( r, \frac{1}{F(k+1)} \right) + N_0 \left( r, \frac{1}{G(k+1)} \right). \]  

(3.14)

Obviously,

\[ \mathcal{N} \left( r, \frac{1}{F(k) - 1} \right) + \mathcal{N} \left( r, \frac{1}{G(k) - 1} \right) \leq N_{E} \left( r, \frac{1}{F(k) - 1} \right) + N \left( r, \frac{1}{F(k) - 1} \right). \]  

(3.15)

Thus, we deduce from (3.8)–(3.10), (3.14) and (3.15) that

\[ T(r, G) \leq (k + 2)\mathcal{N}(r, F) + 2\mathcal{N}(r, G) + N_{k+1} \left( r, \frac{1}{F} \right) + N_{k+1} \left( r, \frac{1}{G} \right) + \mathcal{N} \left( r, \frac{1}{F} \right) + \mathcal{N} \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G). \]  

(3.16)
Note that \( l = 1 \), from Lemma 1, we have
\[
N_k \left( r, \frac{1}{F(k) - 1} \right) \leq N \left( r, \frac{1}{F(k+1)} \right) = N_1 \left( r, \frac{1}{F(k+1)} \right) \leq N_{k+2} \left( r, \frac{1}{F} \right) + (k+1)N(r, F) + S(r, F).
\]

(3.17)

The inequality (3.16) together with (3.17) yields
\[
T(r, G) \leq (2k + 3)N(r, F) + 2N(r, G) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right)
\]
\[
+ N_{k+1} \left( r, \frac{1}{F} \right) + N_{k+1} \left( r, \frac{1}{G} \right) + N_{k+2} \left( r, \frac{1}{F} \right) + S(r, F) + S(r, G).
\]

Hence
\[
T(r, G) \leq [(2k + 3)(1 - \Theta(\infty, F)) + 2(1 - \Theta(\infty, G)) + (1 - \Theta(0, F)) + (1 - \Theta(0, G))
\]
\[
+ (1 - \delta_{k+1}(0, F)) + (1 - \delta_{k+1}(0, G)) + (1 - \delta_{k+2}(0, F)) + \varepsilon]T(r, G) + S(r, G),
\]
for \( r \in I \) and \( 0 < \varepsilon < \Delta_2 - (2k + 9) \), that is
\[
[\Delta_2 - (2k + 9) - \varepsilon]T(r, G) \leq S(r, G),
\]
i.e.,
\[
\Delta_2 \leq 2k + 9,
\]
(3.18)

where
\[
\Delta_2 = (2k + 3)\Theta(\infty, F) + 2\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G) + \delta_{k+2}(0, F)
\]
\[
\geq (2k + 3 + 2) \frac{n - 1}{n} + 2 \cdot \frac{n - 1}{n} + 2 \cdot \frac{n - (k + 1)}{n} + \frac{n - (k + 2)}{n}
\]
\[
= 2k + 10 - \frac{5k + 11}{n}.
\]

Note that (3.18), we have \( n \leq 5k + 11 \), which is in contradiction with hypothesis \( n > 5k + 11 \).

If \( l = 0 \), i.e., \( F^{(k)} \) and \( G^{(k)} \) share 1 IM, at this circumstance, we have
\[
N(r, \varphi) \leq N(r, F) + N(r, G) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right)
\]
\[
+ N_L \left( r, \frac{1}{F^{(k)} - 1} \right) + N_L \left( r, \frac{1}{G^{(k)} - 1} \right) + N_0 \left( r, \frac{1}{F^{(k+1)}} \right) + N_0 \left( r, \frac{1}{G^{(k+1)}} \right).
\]

(3.19)

From Lemma 5, we have
\[
N_L \left( r, \frac{1}{F^{(k)} - 1} \right) + 2N_L \left( r, \frac{1}{G^{(k)} - 1} \right)
\]
\[
\leq N(r, F) + 2N(r, G) + N \left( r, \frac{1}{F^{(k)}} \right) + 2N \left( r, \frac{1}{G^{(k)}} \right) + S(r, F) + S(r, G).
\]

(3.20)

From Lemma 1, we can deduce that
\[
N \left( r, \frac{1}{F^{(k)}} \right) + 2N \left( r, \frac{1}{G^{(k)}} \right) = N_1 \left( r, \frac{1}{F^{(k)}} \right) + 2N_1 \left( r, \frac{1}{G^{(k)}} \right)
\]
\[
\leq N_{k+1} \left( r, \frac{1}{F} \right) + 2N_{k+1} \left( r, \frac{1}{G} \right) + kN(r, F) + 2kN(r, G) + S(r, F) + S(r, G).
\]

(3.21)
When \( l = 0 \), we can get
\[
N \left( r, \frac{1}{F^{(k)}(r)} \right) + N \left( r, \frac{1}{G^{(k)}(r)} \right) \leq N^{l_1}_E \left( r, \frac{1}{F^{(k)}(r)} \right) + N \left( r, \frac{1}{G^{(k)}(r)} \right).
\]

From (3.8)–(3.10) and (3.19)–(3.21) and the above inequality, we can obtain
\[
T(r, G) \leq (2k + 3)N(r, F) + (2k + 4)N(r, G) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right)
+ 2N_{k+1} \left( r, \frac{1}{F} \right) + 3N_{k+1} \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G).
\]

In the same way, we can also get
\[
T(r, G) \leq [(2k + 3)(1 - \Theta(\infty, F)) + (2k + 4)(1 - \Theta(\infty, G)) + (1 - \Theta(0, F)) + (1 - \Theta(0, G))
+ 2(1 - \delta_{k+1}(0, F)) + 3(1 - \delta_{k+1}(0, G)) + \varepsilon]T(r, G) + S(r, G),
\]
for \( r \in I \) and \( 0 < \varepsilon < \Delta_3 - (4k + 13) \), that is
\[
[\Delta_3 - (4k + 13) - \varepsilon]T(r, G) \leq S(r, G),
\]
i.e.,
\[
\Delta_3 \leq 4k + 13, \quad (3.22)
\]
where
\[
\Delta_3 = (2k + 3)\Theta(\infty, F) + (2k + 4)\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + 2\delta_{k+1}(0, F) + 3\delta_{k+1}(0, G)

\geq (4k + 7) \cdot \frac{n - 1}{n} + 2 \cdot \frac{n - 1}{n} + 5 \cdot \frac{n - (k + 1)}{n}

= 4k + 14 - \frac{9k + 14}{n}.
\]

Note that (3.22), we have \( n \leq 9k + 14 \), which is in contradiction with hypothesis \( n > 9k + 14 \).

Hence, we get \( \varphi(z) \equiv 0 \), i.e.,
\[
\frac{F^{(k+2)}}{F^{(k+1)}} - 2 \frac{F^{(k+1)}}{F^{(k)}(r)} = \frac{G^{(k+2)}}{G^{(k+1)}} - 2 \frac{G^{(k+1)}}{G^{(k)}(r)}.
\]

Integration yields
\[
\frac{1}{F^{(k)}(r)} = \frac{bG^{(k)}(r) + a - b}{G^{(k)}(r)} , \quad (3.23)
\]
where \( a \) and \( b \) are constants and \( a \neq 0 \). Obviously, (3.23) means that \( F^{(k)}(r) \) and \( G^{(k)}(r) \) share 1 CM. Next, we consider three cases.

**Case 1.** If \( b = 0 \), then from (3.23), we obtain
\[
F = \frac{1}{a}G + p(z) , \quad (3.24)
\]
where \( p(z) \) is a polynomial.

If \( p(z) \neq 0 \), since \( f \) is transcendental, then by the second fundamental theorem, we have
\[
T(r, F) \leq N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - p} \right) + S(r, F)
\]
\[
\leq N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + S(r, F) . \quad (3.25)
\]
From (3.24), we have
\[ T(r, F) = T(r, G) + S(r, F). \]
Substituting this into (3.25), we get
\[ [\Theta(\infty, F) + \Theta(0, F) + \Theta(0, G) - 2 + \varepsilon]T(r, F) \leq S(r, F), \]
where \( \varepsilon > 0 \). Noticing that
\[ \Theta(\infty, F) + \Theta(0, F) + \Theta(0, G) - 2 \geq 3 \cdot \frac{n-1}{n} - 2 = \frac{n-3}{n} > 0, \]
since \( n > 3k + 8 \). Thus, we have \( T(r, F) = S(r, F) \), which is a contradiction. Therefore, we deduce that \( p(z) \equiv 0 \), that is
\[ F = \frac{1}{a} G. \] (3.26)
Differentiation on both sides of (3.26) \( k \) times yields
\[ F^{(k)} = \frac{1}{a} G^{(k)}. \]
The above equation together with the fact that \( F^{(k)} \) and \( G^{(k)} \) share 1 CM yields \( a = 1 \). Hence (3.26) becomes \( F = G \), that is \( f^n(z) = g^n(z) \), which is equivalent to \( f = tg \), where \( t \) is a constant such that \( t^n = 1 \).

**Case 2.** Suppose that \( b \neq 0 \) and \( a \neq b \).

If \( b = -1 \), then from (3.23), we have
\[ F^{(k)} = \frac{-a}{G^{(k)} - a - 1}. \]
Therefore
\[ N \left( r, \frac{1}{G^{(k)} - a - 1} \right) = N(r, F^{(k)}) = N(r, F). \]

From Lemma 3, we have
\[ T(r, G) \leq N(r, G) + N_{k+1} \left( r, \frac{1}{G} \right) + N \left( r, \frac{1}{G^{(k)} - (a + 1)} \right) - N_0 \left( r, \frac{1}{G^{(k+1)}} \right) + S(r, G) \]
\[ \leq N(r, G) + N_{k+1} \left( r, \frac{1}{G} \right) + N(r, F) + S(r, G) \]
\[ \leq (k + 2)N(r, G) + 2N(r, G) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) \]
\[ + N_{k+1} \left( r, \frac{1}{F} \right) + N_{k+1} \left( r, \frac{1}{G} \right) + S(r, G). \]

Hence
\[ T(r, G) \leq [(k + 2)(1 - \Theta(\infty, F)) + 2(1 - \Theta(\infty, G)) + (1 - \Theta(0, F)) + (1 - \Theta(0, G)) \]
\[ + (1 - \delta_{k+1}(0, F)) + (1 - \delta_{k+1}(0, G)) + \varepsilon]T(r, G) + S(r, G), \]
for \( r \in I \) and \( 0 < \varepsilon < \Delta_1 - (k + 7) \), that is
\[ [\Delta_1 - (k + 7) - \varepsilon]T(r, G) \leq S(r, G), \]
i.e.,
\[ \Delta_1 \leq k + 7, \tag{3.27} \]
where
\[
\Delta_1 = (k + 2) \Theta(\infty, F) + 2 \Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G)
\geq (k + 4) \frac{n - 1}{n} + 2 \cdot \frac{n - 1}{n} + 2 \cdot \frac{n - (k + 1)}{n}
= k + 8 - \frac{3k + 8}{n} > k + 7,
\]
since \( n > 3k + 8 \). This contradicts that (3.27).
If \( b \neq -1 \), from (3.23) we obtain that
\[
F^{(k)} - (1 + 1/b) = \frac{-a}{b^2[G^{(k)} + (a - b)/b]}.
\]
Therefore
\[
\overline{N}(r, \frac{1}{G^{(k)} + (a - b)/b}) = \overline{N}(r, F^{(k)} - (1 + 1/b)) = \overline{N}(r, F).
\]
From Lemma 3, we have
\[
T(r, G) \leq \overline{N}(r, G) + N_{k+1}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G^{(k)} + a/b}\right) - N_0\left(r, \frac{1}{G^{(k+1)}}\right) + S(r, G)
\leq \overline{N}(r, G) + N_{k+1}\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + S(r, G)
\leq (k + 2)\overline{N}(r, F) + 2\overline{N}(r, G) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right)
+ N_{k+1}\left(r, \frac{1}{F}\right) + N_{k+1}\left(r, \frac{1}{G}\right) + S(r, G).
\]
Using the argument as in the state when \( b = -1 \), we can also get a contradiction.

Case 3. Suppose that \( b \neq 0 \) and \( a = b \).
If \( b \neq -1 \), then from (3.20), we have
\[
\frac{1}{F^{(k)}} = \frac{bG^{(k)}}{(1+b)G^{(k)}-1}.
\]
Hence,
\[
\overline{N}\left(r, \frac{1}{G^{(k)} - 1/(1+b)}\right) = \overline{N}\left(r, \frac{1}{F^{(k)}}\right) \leq \overline{N}\left(r, \frac{F}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F}\right)
\leq T\left(r, \frac{F}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F}\right) \leq T\left(r, \frac{F^{(k)}}{F}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, F)
\leq N\left(r, \frac{F^{(k)}}{F}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, F)
\leq (k + 1)\overline{N}\left(r, \frac{1}{F}\right) + k\overline{N}(r, F) + S(r, F).
\]
From Lemma 3, we get
\[
T(r, G) \leq \overline{N}(r, G) + N_{k+1}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G^{(k)} - 1/(b+1)}\right) - N_0\left(r, \frac{1}{G^{(k+1)}}\right) + S(r, G)
\]
\[
\leq N(r, G) + N_{k+1} \left( r, \frac{1}{G} \right) + N \left( r, \frac{1}{F(k)} \right) + S(r, G)
\]

\[
\leq kN(r, F) + N(r, G) + (k+1)N \left( r, \frac{1}{F} \right) + N_{k+1} \left( r, \frac{1}{G} \right) + S(r, G).
\]

Using the argument as in Case 2, a contradiction can also be obtained.

If \( b = -1 \), then (3.23) implies

\[ F^{(k)}(z)G^{(k)}(z) \equiv 1. \]

That is

\[ [f^n(z)]^{(k)}[g^n(z)]^{(k)} \equiv 1. \tag{3.28} \]

First, we prove that

\[ f \neq 0, \infty, \quad g \neq 0, \infty. \tag{3.29} \]

Suppose that \( f(z) \) has a zero \( z_0 \) of order \( p \), then \( z_0 \) is a zero of \([f^n(z)]^{(k)}\) of order \((3k + k_1)p - k\) and \( z_0 \) must be a pole of \([g^n(z)]^{(k)}\) of order \((3k + k_1)q + k\), where \( k_1 > 8 \). From (3.28), we get

\[ (3k + k_1)p - k = (3k + k_1)q + k, \]

i.e.,

\[ (3k + k_1)(p - q) = 2k, \]

which is impossible since \( p \) and \( q \) are integers and \( k_1 > 8 \). Therefore \( f \neq 0, g \neq 0 \). Similarly, \( f \neq \infty, \ g \neq \infty \).

Hence (3.29) holds.

From (3.28) and (3.29), we get

\[ [f^n(z)]^{(k)} \neq 0, \quad [g^n(z)]^{(k)} \neq 0. \tag{3.30} \]

From (3.28)–(3.30) and Lemma 4, we get for \( k \geq 2 \) that \( f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \((-1)^k(c_1c_2)^n(nc)^{2k} = 1\).

Next, we consider \([f^n(z)]^{(k)}[g^n(z)]^{(k)} \equiv 1\) for the case \( k = 1 \). That is

\[ n^2 f^{n-1}f'g^{n-1}g' \equiv 1. \tag{3.31} \]

Now, we prove that

\[ f \neq 0, \infty, \quad g \neq 0, \infty. \tag{3.32} \]

In fact, suppose that \( f \) has a zero \( z_0 \) with order \( p \). Then \( z_0 \) is a pole of \( g \)(with order \( q \), say), by (3.31), we get

\[(n - 1)p + p - 1 = (n - 1)q + q + 1, \]

i.e., \( n(p - q) = 2 \), which is impossible since \( p \) and \( q \) are integers and \( n > 3k + 8 > 11 \). Therefore \( f \neq 0, g \neq 0 \). Similarly, \( f \neq \infty, g \neq \infty \). Hence (3.32) holds.

Thus, there exist two entire functions \( \alpha(z) \) and \( \beta(z) \) such that

\[ f(z) = e^{\alpha(z)}, \quad g(z) = e^{\beta(z)}. \tag{3.33} \]

Inserting (3.33) into (3.31), we get

\[ n^2 \alpha' \beta' e^{n(\alpha + \beta)} \equiv 1. \tag{3.34} \]

Thus \( \alpha' \) and \( \beta' \) are entire functions which have no zeros, and we may set

\[ \alpha' = e^{\delta(z)}, \quad \beta' = e^{\gamma(z)}, \tag{3.35} \]

where \( \delta(z) \) and \( \gamma(z) \) are entire functions. Then (3.34) becomes

\[ n^2 e^{n(\alpha + \beta) + \delta + \gamma} \equiv 1. \]
Differentiating the above equation gives
\[ n(\alpha' + \beta') + \delta' + \gamma' \equiv 0. \]
i.e.,
\[ n(\epsilon^\delta + \epsilon'^\delta) + \delta' + \gamma' \equiv 0. \] (3.36)
If \( \delta' + \gamma' \equiv 0 \), then \( \epsilon^\delta + \epsilon'^\delta \equiv 0 \), i.e., \( \delta = \gamma + (2m + 1)\pi i \) for some integer \( m \).
Inserting this in (3.36), we deduce that \( \delta' = \gamma' = 0 \), and so \( \delta \) and \( \gamma \) are constants, i.e., \( \alpha' \) and \( \beta' \) are constants.
From this and (3.33) and (3.34), we get \( f(z) = c_1 e^{cz}, \) \( g(z) = c_2 e^{-cz}, \) where \( c_1, c_2 \) and \( c \) are three constants satisfying \( (c_1 c_2)^n (n c)^{2k} = -1 \).
If \( \delta' + \gamma' \not\equiv 0 \), it follows by (3.36) that
\[ \frac{n}{\delta' + \gamma'} \epsilon^\delta + \frac{n}{\delta' + \gamma'} \epsilon'^\delta \equiv -1. \]
Set
\[ g_1 = \frac{n}{\delta' + \gamma'} \epsilon^\delta, \quad g_2 = \frac{n}{\delta' + \gamma'} \epsilon'^\delta, \] (3.37)
so \( g_1 + g_2 = -1 \) and
\[ T(r, g_1) = \overline{N}(r, g_1) + \overline{N} \left( r, \frac{1}{g_1} \right) + \overline{N} \left( r, \frac{1}{g_2} \right) + S(r, g_1) \]
\[ = \overline{N} \left( r, \frac{1}{\delta' + \gamma'} \right) + S(r, g_1) \]
\[ = o[T(r, \epsilon^\delta) + T(r, \epsilon'^\delta)] + S(r, g_1). \] (3.38)
By symmetry, we have
\[ T(r, g_2) = \overline{N}(r, g_2) + \overline{N} \left( r, \frac{1}{g_1} \right) + \overline{N} \left( r, \frac{1}{g_2} \right) + S(r, g_2) \]
\[ = o[T(r, \epsilon^\delta) + T(r, \epsilon'^\delta)] + S(r, g_2). \] (3.39)
Obviously, (3.37) gives
\[ T(r, g_1) = T(r, \epsilon^\delta) + o[T(r, \epsilon^\delta) + T(r, \epsilon'^\delta)], \]
\[ T(r, g_2) = T(r, \epsilon'^\delta) + o[T(r, \epsilon^\delta) + T(r, \epsilon'^\delta)], \]
\[ S(r, g_1) + S(r, g_2) = o[T(r, \epsilon^\delta) + T(r, \epsilon'^\delta)]. \]
So with the help of (3.38) and (3.39), we obtain
\[ T(r, \epsilon^\delta) + T(r, \epsilon'^\delta) = o[T(r, \epsilon^\delta) + T(r, \epsilon'^\delta)], \]
which is impossible.
Therefore for all \( k \geq 1 \), we get \( f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}, \) where \( c_1, c_2 \) and \( c \) are three constants satisfying \( (-1)^k (c_1 c_2)^n (n c)^{2k} = 1 \).
The proof of Theorem 1 has been completed.

Remark. For the case \( k = 1 \), let \( h = \frac{1}{2g} \). By (3.31) and using the same reasoning as, in the proof of Theorem 2 (see [6, p. 401-402]), we can also obtain the desired result.
Using the same method as Theorem 1, we can prove Theorem 2.

Acknowledgements

The authors would like to thank the referees for their valuable comments, corrections and suggestions.
References