Special Finsler Spaces Admitting Metric Like Tensor Field

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Abstract

In this work we modify the special Finsler spaces like C-reducible, semi-C-reducible, quasi-C-reducible are admitting the tensor field $X_{hk} = h_{hk} + X_{00}^l h_l k$, which satisfies the condition $C^h_{ij} X_{hk} = C^h_{ijk}$. Similarly, we have also worked out for S3-like, $C^h$-recurrent, P-reducible and T-conditions of Finsler spaces.

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1 Introduction

The terminology and notations are referred to [1], [4] and [6]. Let $F^n = (M^n, L)$ be a Finsler space on a differentiable manifold $M$ endowed with a fundamental function $L(x, y)$. We use the following notations: [4][6]

\begin{align*}
a) \quad g_{ij} &= \frac{1}{2} \dot{\theta}_i \dot{\theta}_j L^2, \quad \dot{\theta}_i = \frac{\partial}{\partial y^i}, \\
b) \quad C^i_{ijk} &= \frac{1}{2} \hat{\theta}_k g_{ij},
\end{align*}
There are three kinds of torsion tensors in Cartan’s theory of Finsler spaces. Two of them are $h(h\nu)$-torsion tensor $C_{ijk}$ and $(\nu)h\nu$-torsion tensor $P_{ijk}$, which are symmetric in all their indices. The contravariant components of $(\nu)h\nu$-torsion tensor is given by $C_{hk}^{ij} = g_{hk}C_{ijk}$, which may be treated as Christoffel symbols of second kind of each tangent Riemannian space of Finsler space $F^n$. Here, $g_{hk}$ is the inverse of metric tensor $g_{hk}$ of $F^n$. If $l_i$ is the normalized element of support $h_{ij}$ is the angular metric tensor given by $h_{ij} = g_{ij} - l_i l_j$, then $C_{hk}^{ij}g_{hk} = C_{ijk} = C_{hk}^{ij}h_{hk}$.

If $b_i$ are components of a concurrent vector field, then $b_{ij} = -g_{ij}$ and $b_{iij} = 0$, where $/j$ and $|j$ denote the $h$ and $\nu$-covariant derivatives with respect to Cartan’s connection $CT$. From this it follows that $b_i$ are functions of position only, and $C_{ij}^h b_h = 0$. Thus if we consider a tensor field is given by $B_{ij} = g_{ij} + \alpha l_i l_j + \beta b_i b_j$, where $\alpha$ and $\beta$ are scalar functions, then $C_{ij}^h B_{hk} = C_{ijk}$.

The purpose of the present paper is to study the existence of any symmetric covariant tensor $X_{hk}$ which satisfies $X_{hk} = h_{hk} + X_{00} l_h l_k$. Throughout the paper we are concerned with non-Riemannian Finsler space having positive definite metric tensor $g_{ij}$. From (4) we have, $C^h X_{hk} = C_k$ and $C_{ij}^h X_{h0} = 0$, where $C^h = C_{ij}^h g^{ij}$ and 0 denotes the contraction with $l^i$.

2 The Existence Of Covariant Tensor $X_{hk}$ In C-Reducible Finsler Space:

In a C-reducible Finsler space the $(h)h\nu$-torsion tensor $C_{ij}^h$ is given by [2][5]

$$C_{ij}^h = (C^h h_{ij} + C_i h_j^h + C_j h_i^h)/(n+1).$$
Now contracting above equation by $X_{hk}$, then from equations (4) and (1)(d), we have

\[C^h_{ij} X_{hk} = X_{hk}(C^h_{ij} + C_i h_j^h + C_j h_i^h)/(n + 1),\]
\[C^h_{ij} X_{hk} = 1/(n + 1)[C^h_{ij}(h_{kk} + X_{00}l_h l_k) + C_i h_j^h(h_{kk} + X_{00}l_h l_k) + C_j h_i^h(h_{kk} + X_{00}l_h l_k)]\]
\[C^h_{ij} X_{hk} = 1/(n + 1)[C_k h_{ij} + C_i h_{jk} + C_i h_j^h X_{00}l_h l_k + C_j h_{ik} + C_j h_i^h X_{00}l_h l_k]\]
\[C^h_{ij} X_{hk} = 1/(n + 1)[C_k h_{ij} + C_i h_{jk} + C_j X_{00}l_h l_k(\delta_j^h - l^h l_j)]\]
\[C^h_{ij} X_{hk} = 1/(n + 1)[C_k h_{ij} + C_i h_{jk} + C_j X_{00}l_h l_k(l_j - l_i)]\]
\[C^h_{ij} X_{hk} = C_{ijk} . \tag{7}\]

**Theorem 2.1** In a C-reducible Finsler space the covariant tensor field $X_{hk}$ satisfies (4) is of the form (7).

Consider a Semi-C-reducible Finsler space $C^h_{ij}$ is given by [3],

\[C^h_{ij} = (C^h_{ij} + C_i h_j^h + C_j h_i^h)p/(n + 1) + (C^h_{ij} q/C^2). \tag{8}\]

Now contracting above equation by $X_{hk}$ and using equations (4) and (1)(d), we have

\[C^h_{ij} X_{hk} = X_{hk}(C^h_{ij} + C_i h_j^h + C_j h_i^h)p/(n + 1) + X_{hk}(C^h_{ij} q/C^2),\]
\[C^h_{ij} X_{hk} = (C^h_{ij}(h_{kk} + X_{00}l_h l_k) + C_i h_j^h(h_{kk} + X_{00}l_h l_k) + C_j h_i^h(h_{kk} + X_{00}l_h l_k))\]
\[p/(n + 1) + (C^h_{ij} q/C^2),\]
\[C^h_{ij} X_{hk} = (C_k h_{ij} + C_i h_{jk} + C_j h_{ik})p/(n + 1) + (C_k C_j) q/C^2,\]
\[C^h_{ij} X_{hk} = C_{ijk} . \tag{9}\]

**Theorem 2.2** In a Semi-C-reducible Finsler space the tensor field $X_{hk}$ satisfies (4) is of the form (9).

Consider a Quasi-C-reducible Finsler space $C^h_{ij}$ is given by [3],

\[C^h_{ij} = (C^h_{ij} + C_i A_j^h + C_j A_i^h). \tag{10}\]

Now contracting above equation by $X_{hk}$ and using equations (4) and (1)(d), we have

\[C^h_{ij} X_{hk} = (C^h_{ij} + C_i A_j^h + C_j A_i^h) X_{hk},\]
\[C^h_{ij} X_{hk} = (C^h_{ij}(h_{kk} + X_{00}l_h l_k) + C_i A_j^h(h_{kk} + X_{00}l_h l_k)\]
\[+ C_j A_i^h(h_{kk} + X_{00}l_h l_k),\]
\[C^h_{ij} X_{hk} = (C_k A_{ij} + C_i A_{jk} + C_j A_{ik}),\]
\[C^h_{ij} X_{hk} = C_{ijk} . \tag{11}\]

**Theorem 2.3** In a Quasi-C-reducible Finsler space the tensor field $X_{hk}$ satisfies (4) is of the form (11) if $A^h_{lh} = 0$. 

3 The Existence Of Covariant Tensor $X_{hk}$ In S3-like Finsler Space:

In a S3-like Finsler space, whose $\nu-$ curvature tensor of cartons connection CT is given by [5],

$$L^2 S_{ihk}^m = S(h_{ih} h_k^m - h_{ik} h_h^m).$$  \hspace{1cm} (12)

Contacting above equation by $X_{mj}$ and using equations (4) and (1)(d), we have

\[ L^2 S_{ihk}^m X_{mj} = S(h_{ih} h_k^m X_{mj} - h_{ik} h_h^m X_{mj}), \]
\[ L^2 S_{ihk}^m X_{mj} = S[h_{ih} h_k^m (h_{mj} - X_{00} l_m l_j) - h_{ik} h_h^m (h_{mj} - X_{00} l_m l_j)], \]
\[ L^2 S_{ihk}^m X_{mj} = S[h_{ih} h_{jk} - X_{00} h_{ik} h_h^m l_m l_j - h_{ik} h_{hj} + X_{00} h_{ik} h_h^m l_m l_j], \]
\[ L^2 S_{ihk}^m X_{mj} = S[h_{ih} h_{jk} - h_{ik} h_{hj}], \]
\[ L^2 S_{ihk}^m X_{mj} = S h_{ijk}. \] \hspace{1cm} (13)

**Theorem 3.1** In a S3-like Finsler space, the covariant tensor field $X_{mj}$ satisfies (4) is of the form (13).

Next we consider S4-like Finsler space, whose $\nu$-curvature tensor of cartons connection CT is given by [7],

$$L^2 S_{ihk}^m = h_h^m M_{ik} + h_{ik} M_h^m - h_{hk} M_i^m - h_i^m M_{hk}. \hspace{1cm} (14)$$

Contacting above equation by $X_{mj}$ and using equations (4) and (1)(d), we have

\[ L^2 S_{ihk}^m X_{mj} = h_h^m M_{ik} X_{mj} + h_{ik} M_h^m X_{mj} - h_{hk} M_i^m X_{mj} - h_i^m M_{hk} X_{mj}, \]
\[ L^2 S_{ihk}^m X_{mj} = h_h^m M_{ik} X_{mj} + h_{ik} M_h^m X_{mj} - h_{hk} M_i^m X_{mj} - h_i^m M_{hk} X_{mj}, \]
\[ L^2 S_{ihk}^m X_{mj} = L^2 S_{hijk}. \] \hspace{1cm} (15)

**Theorem 3.2** In a S4-like Finsler space, the covariant tensor field $X_{ij}$ satisfies (4) is of the form (15).

Next we consider a $\nu$-curvature tensor of Cartons connection CT is given by [2],

$$S_{ihk}^m = C_{hr}^m C_{ik}^r - C_{kr}^m C_{ih}^r. \hspace{1cm} (16)$$

Contacting above equation by $X_{mj}$ and using equations (4) and (1)(d), we have

\[ S_{ihk}^m X_{mj} = C_{hr}^m C_{ik}^r X_{mj} - C_{kr}^m C_{ih}^r X_{mj}, \]
\[ S_{ihk}^m X_{mj} = C_{hr}^m C_{ik}^r (h_{mj} + X_{00} l_m l_j) - C_{kr}^m C_{ih}^r (h_{mj} + X_{00} l_m l_j), \]
\[ S_{ihk}^m X_{mj} = C_{hjr}^r C_{ik}^r - C_{krj}^r C_{ih}^r, \]
\[ S_{ihk}^m X_{mj} = S_{hijk}. \] \hspace{1cm} (17)
Theorem 3.3  In a $\nu$-curvature tensor, the covariant tensor field $X_{mj}$ satisfies (4) is of the form (17).

Now we concerned with a space of scalar curvature in Berwald’s sense. It is characterized by the equation is \[ R_{ij} = h_{ij} k_j - h_{jk} k_i. \] (18)

where $h_{ij}$ is the angular metric tensor and the scalar curvature $K$ is a function scalar field.

Contacting above equation by $X_{il}$ and using equations (4) and (1)(d), we get

\[
\begin{align*}
R_{ij} X_{il} &= h_{ik} K_j X_{il} - h_{jk} K_i X_{il}, \\
R_{ij} X_{il} &= h_{ik} K_j (h_{il} + X_{00} l_i l_l) - h_{jk} K_i (h_{il} + X_{00} l_i l_l), \\
R_{ij} X_{il} &= K_j h_{ki} - K_k h_{ji}, \\
R_{ij} X_{il} &= R_{ijk}.
\end{align*}
\] (19)

Theorem 3.4  In a space of scalar curvature tensor, the covariant tensor field $X_{il}$ satisfies (4) is of the form (19).

4 The Existence Of Covariant Tensor $X_{hk}$ In P-Reducible Finsler Space:

The P-reducible Finsler space is given as [5],

\[
P_{jk}^m = (h_{jk} P_k + h_{jk} P^m + h_{jk} P_j)/(n + 1),
\] (20)

Contacting above equation by $X_{mi}$ and using equations (4) and (1)(d), we have

\[
\begin{align*}
P_{jk}^m X_{mi} &= (h_{jk} P_k X_{mi} + h_{jk} P^m X_{mi} + h_{jk} P_j X_{mi})/(n + 1), \\
P_{jk}^m X_{mi} &= (h_{jk} P_k (h_{mi} + X_{00} l_m l_i) + h_{jk} P^m (h_{mi} + X_{00} l_m l_i) + h_{jk} P_j (h_{mi} + X_{00} l_m l_i))/(n + 1), \\
P_{jk}^m X_{mi} &= (h_{jk} P_k + h_{jk} P_i + h_{jk} P_j)/(n + 1), \\
P_{jk}^m X_{mi} &= P_{ijk}.
\end{align*}
\] (21)

Theorem 4.1  In a P-reducible Finsler space, the covariant tensor field $X_{mi}$ satisfies (4) is of the form (21).

5 The Existence Of Covariant Tensor $X_{hk}$ In $C^h$-Recurrent Finsler Space:

Now we consider a $C^h$-recurrent Finsler space is given as [2],

\[
C_{jk/n}^m = \alpha h C_{jk}^m.
\] (22)
Contacting above equation by $X_{mi}$ and using equations (4) and (1)(d), we can written as

$$C^m_{jk/h}X_{mi} = \alpha_h C^m_{jk}X_{mi},$$
$$C^m_{jk/h}X_{mi} = \alpha_h C^m_{jk}(h_{mi} + X_{00l}l_i),$$
$$C^m_{jk/h}X_{mi} = \alpha_h C_{ijk},$$
$$C^m_{jk/h}X_{mi} = C_{ijk/h}. \quad (23)$$

**Theorem 5.1** In a $C^h$-recurrent Finsler space, the covariant tensor field $X_{mi}$ satisfies (4) is of the form (23).

6 The Existence Of Covariant Tensor $X_{hk}$ In T-Condition :

Finsler space satisfying T-condition can be defined as,[6]

$$T^m_{ijk} = L^m_{ij/k} + l^m_{c_{ijk}} + l^m_{c_{ikj}} + l^m_{c_{ijk}} = 0, \quad (24)$$

Contacting above equation by $X_{hm}$ and using equations (4) and (1)(d), we have

$$T^m_{ijk}X_{hm} = L^m_{ij/k}X_{hm} + l^m_{c_{ijk}}X_{hm} + l^m_{c_{ikj}}X_{hm} + l^m_{c_{ijk}}X_{hm} + l^m_{c_{ijk}}X_{hm} = 0,$$
$$T^m_{ijk}X_{hm} = L^m_{ij/k}X_{hm} + l^m_{c_{ijk}}(h_{hm} + X_{00l}l_m) + l^m_{c_{ikj}}(h_{hm} + X_{00l}l_m) +$$
$$l^m_{c_{ijk}}(h_{hm} + X_{00l}l_m) + l^m_{c_{ijk}}(h_{hm} + X_{00l}l_m) = 0,$$
$$T^m_{ijk}X_{hm} = L^m_{ij/k} + h^m_{c_{ijk}} + X_{00l}l_m l^m_{c_{ijk}} + l^m_{c_{ijk}} + l^m_{c_{ijk}} + l^m_{c_{ijk}} +$$
$$l^m_{c_{ijk}} + X_{00l}l_m l^m_{c_{ijk}} = 0,$$
$$T^m_{ijk}X_{hm} = T_{hijk} + X_{00l}c_{ijk} = 0,$$
$$T^m_{ijk}X_{hm} = T_{hijk} = 0. \quad (25)$$

**Theorem 6.1** If the Finsler space satisfying T-condition, then the covariant tensor field $X_{hm}$ satisfies (4) is of the form (25) provided $X_{00l}c_{ijk} = 0$.

Finsler space satisfying generalized T-condition can be defined as [5],

$$T^h_j = L^h_{ij} + l^h_{C_{ij}} + l^h_{C_{ij}} = 0. \quad (26)$$

Contacting above equation by $X_{ih}$ and using equations (4) and (1)(d), we obtain

$$T^h_jX_{ih} = L^h_{ij}X_{ih} + l^h_{C_{ij}}X_{ih} + l^h_{C_{ij}}X_{ih} = 0,$$
$$T^h_jX_{ih} = L^h_{ij}(h_{ih} + X_{00l}l_i) + l^h_{C_{ij}}(h_{ih} + X_{00l}l_i) + l^h_{C_{ij}}(h_{ih} + X_{00l}l_i) = 0,$$
$$T^h_jX_{ih} = L^h_{ij} + l^h_{C_{ij}} + l^h_{C_{ij}} = 0,$$
$$T^h_jX_{ih} = T_{ij}. \quad (27)$$
Theorem 6.2 If the Finsler space satisfying generalized T-Condition, then the covariant tensor field $X_{ih}$ satisfies (4) is of the form (27).

References


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