SOME PARAMETERS ON NEIGHBORHOOD NUMBER OF A GRAPH

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Abstract

A set $S \subseteq V$ is a neighborhood set of G , if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by v and all vertices adjacent to v . The neighborhood number $\eta(G)$ of G is the minimum cardinality of a neighborhood set of G . In this paper, we extended the concept of neighborhood number and its relationship with other related parameters are explored.

Keywords: Graph, neighborhood set, neighborhood number.

1 Introduction

All the graph considered here are finite, undirected and connected with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges at a graph G , respectively. In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices X and $N(v)$ and $N[v]$ denote

¹ Thanks are due to Prof. M.I.Jinnah for his help and valuable suggestions in the preparation of this paper.

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the open and closed neighborhoods of a vertex v , respectively. Let $\deg(v)$ be the degree of vertex v and as usual $\delta(G)$, the minimum degree and $\Delta(G)$, the maximum degree of a graph G . $\alpha_0(G)(\alpha_1(G))$, is the minimum number of vertices (edges) in a vertex (edge) cover of G . $\beta_0(G)(\beta_1(G))$, is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G . For any undefined term in this paper, we refer the reader to Harary [4]. A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G , [5]. In 1985, Sampathkumar and Neeralagi [10] introduced the concept of the neighborhood number of a graph, as follows. A set $S \subseteq V$ is a neighborhood set of G , if $G = \bigcup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by v and all vertices adjacent to v . The neighborhood number $\eta(G)$ of G is the minimum cardinality of a neighborhood set of G . A graph G having k - disjoint neighborhood set (kDN-set) with $k \geq 2$ is called a k -disjoint neighborhood graph (abbreviated kDN-graph), where k is a positive integer. In fact, if $k = 2$, then G having a 2-disjoint neighbourhood set (2DN-set). The dual neighborhood number $\eta^{+2}(G) = \text{Min.}\{|S_1| + |S_2| : S_1, S_2 \text{ are 2DN-set of } G\}$, [7] & [9]. Further, a neighborhood set $S \subseteq V$ is called an independent neighborhood set, if $\langle S \rangle$ is an independent and neighborhood set of G , [8] /paired neighborhood set, if $\langle S \rangle$ contains at least one perfect matching of G , [6] & [11] /maximal neighborhood set, if $V - S$ does not contain a neighborhood set of G , [12]/inverse neighborhood set, if $V - S$ contain a neighborhood set of G , [2]/ dual neighborhood set, if union of minimum 2DN-set of G , [3]. The minimum cardinality taken over all independent/maximal / inverse/dual neighborhood set in G is called an independent/paired /maximal / inverse/dual neighborhood number of G and is denoted by $\eta_i(G) / \eta_{pr}(G) / \eta_m(G) / \eta^{-1}(G) / \eta^{+2}(G)$, respectively. A neighborhood set S with minimum cardinality is called η - set. Similarly, the other sets can be expected. For more details on neighbourhood sets, we refer [1] & [2].

We make use of the following results in sequel.

Theorem 1.1 [4] *A graph is bipartite if and only if all its cycles are even*

Theorem 1.2 [10]

- (i) *Let G be any graph and S be any subset of $V(G)$. Then S is an η -set of G if and only if every edge in $\langle V - S \rangle$ belongs to $\langle N[u] \rangle$ for some $u \in S$.*
- (ii) *If G has no triangles, then $\eta(G) = \alpha_0(G)$.*

Theorem 1.3 [12] *A neighborhood set S of G is a maximal neighborhood set of G if and only if there exist two adjacent vertices $u, v \in S$ such that every*

vertex $w \in V - S$ is adjacent to at most one of u and v .

2 Preliminary results

Computed values of $\eta_i(G)$ / $\eta_{pr}(G)$ / $\eta_m(G)$ / $\eta_{-1}(G)$ / $\eta^{+2}(G)$ some special classes graphs are stated without proof.

Proposition 2.1 For any Path P_p with $p \geq 2$ vertices,

- (i) $\eta_i(P_p) = p/2$ if p is even and $(p-1)/2$ if p is odd
- (ii) $\eta_{pr}(P_p) = \lceil 2p/3 \rceil$ if $p \equiv -1, 0 \pmod{3}$ and $\lceil 2p/3 \rceil - 1$ if $p \equiv 1 \pmod{3}$
- (iii) $\eta_m(P_p) = (p+1)/2$ if p is odd and $(p+2)/2$ if p is even
- (iv) $\eta^{-1}(P_p) = p - \eta(P_p)$
- (v) $\eta^{+2}(P_p) = p$

Proposition 2.2 For any cycle C_p with $p \geq 3$ vertices,

- (i) $\eta(C_p) = \lceil p/2 \rceil$ if p is even $p \geq 4$
- (ii) $\eta_i(C_p) = \eta^{-1}(C_p)$ if $p = 2n$ and $3n$ with $n \geq 2$
- (iii) $\eta_{pr}(C_p) = \lceil 2p/3 \rceil + 1$ if $p \equiv 1 \pmod{3}$ otherwise $\lceil 2p/3 \rceil$
- (iv) $\eta_m(C_p) = \lceil p/2 \rceil$ if p is odd and $\lceil 2p/3 \rceil$ if p is even
- (v) $\eta^{+2}(C_p) = p$

Proposition 2.3 For any complete bipartite graph $K_{r,s}$ with $1 \leq r \leq s$ vertices,

- (i) $\eta(K_{r,s}) = \eta_i(K_{r,s}) = r$
- (ii) $\eta_{pr}(K_{r,s}) = 2r$ if $1 \leq r \leq s$
- (iii) $\eta_m(K_{r,s}) = r + 1$ if $1 \leq r \leq s$
- (iv) $\eta^{-1}(K_{r,s}) = p - \eta(K_{r,s})$
- (v) $\eta^{+2}(K_{r,s}) = p$.

A graph G for which k -independent neighborhood set (kIN - set) with $k \geq 2$ is called a kIN -graph. Also, here we consider an invariant to both $\eta^{+2}(G)$ and $\eta_i^{+2}(G)$, namely, the minimum cardinality of the disjoint union of minimum neighborhood set S and an independent neighborhood set S_i , which we will denote $\eta\eta_i(G)$. We will call such a pair of neighborhood sets (S, S_i) a $\eta\eta_i$ -pair (or simply, a mixed η - set). We note that every graph G with no isolates has a $\eta\eta_i$ -pair, which can be found by letting S_i be any maximal independent set, and then noting that complement $V - S_i$ is a neighborhood

set, and there fore contains a minimal neighborhood set, say S .

By the definitions of $\eta(G)$ / $\eta_i(G)$ / $\eta_{pr}(G)$ / $\eta_m(G)$ / $\eta^{-1}(G)$ / $\eta^{+2}(G)$, we have the following inequalities, since their proofs are immediate, they are omitted.

Proposition 2.4 *Let G be a 2DN- graph with no isolated vertices. Then,*

- (i) $\gamma \leq \eta(G) \leq \eta_m(G)$,
- (ii) $\eta \leq \eta_i(G) \leq \eta^{+2}(G)$,
- (iii) $2 \leq \eta_{pr}(G) \leq \eta^{+2}(G) \leq p$,
- (iv) $2 \leq \eta^{+2}(G) \leq \eta(G) + \beta_0(G)$,
- (v) $\eta(G) \leq \eta^{-1}(G) \leq p - \eta(G) \leq \eta^{+2}(G)$,
- (vi) $\eta(G) + 1 \leq \eta^{+2}(G) \leq \eta(G) + \eta^{-1}(G)$,
- (vii) $2\eta(G) \leq \eta^{+2}(G) \leq \eta\eta_i(G) + \eta_i^{+2}(G)$.

3 Main results

Theorem 3.1 *For any graph G ,*

- (i) $\{\eta(G), \eta_i(G)\} = p$ if and only if $G = \overline{K_p}$,
- (ii) $\eta_m(G) = p$ if and only if $G = K_p$, provided graph G with no isolated vertices,
- (iii) $\eta_{pr}(G) = p$ if and only if $G = K_{2,2}$, provided G is a connected graph with no isolated vertices,
- (iv) $\eta^{+2}(G) = p$ if and only if G is a bipartite graph, provided G is a connected graph with no isolated vertices.

Proof.

- (i) It is straightforward to check that $\{\eta_i(\overline{K_p}), \eta(\overline{K_p})\} = p$, since G is totally disconnected graph and each component must be a/an neighborhood / independent neighborhood set of G , respectively. This proves necessity. To prove the sufficiency, suppose $G \neq \overline{K_p}$, then there exist at least two vertices u and v such that u and v are adjacent, thus either u or v belongs to neighborhood set of G . This implies that $\{\eta_i(\overline{K_p}), \eta(\overline{K_p})\} < p$, which is a contradiction.
- (ii) &
- (iii) Suppose $\{\eta_m(G), \eta_{pr}(G)\} = p$ holds. On contrary, suppose $G \neq \{K_p / K_{2,2}\}$

then there exist three vertices u, v and w such that u and v are adjacent and w is adjacent to at most one of u and v . This implies that $(V - w)$ is an η_m -set / η_{pr} -set of G , which is a contradiction. Thus the sufficiency is proved. Necessity is easy to check.

- (iv) Clearly, a graph is bipartite if and only if each of its components is bipartite. So, without loss of generality, we assume that G is connected. Let G be a bipartite graph with $V = V_1 \cup V_2$, so that every line of G joins a vertex of V_1 with the vertex of V_2 . Then $\langle V_1 \rangle$ and $\langle V_2 \rangle$ have independent set of $V(G)$, and the minimum and maximum cardinality of V_1 and V_2 have a η -set and η^{-1} -set of G , respectively. Thus $\eta^{+2}(G) = p$. This proves the necessity. Assume that $\eta^{+2}(G) = p$ and G is not a bipartite graph. Then there exist at least three vertices u, v and w such that u and v are adjacent and w is adjacent to both u and v , which is form a odd cycle and by Theorem 1.1, this implies that $(V - w)$ is a $2DN$ -set of G , which is a contradiction. Thus the sufficiency is proved. $\square$

Theorem 3.2 For any graph G with no isolated vertices,

- (i) $p - q + q_0 \leq \eta_i(G) \leq \beta_0(G)$, where $q_0 = \text{Min.}\{q(\langle S \rangle) : S \text{ is a } \eta\text{-set of } G\}$,
- (ii) $p - q + 1 \leq \eta_m(G) \leq \alpha_0(G) + 1$,
- (iii) $p/\Delta(G) \leq \eta_{pr}(G) \leq 2\beta_1(G)$,
- (iv) $(4p - 2q)/3 \leq \eta^{+2}(G) \leq 2\beta_1(G) + 1$, provided G is kDN -graph with $\delta(G) \geq 2$ in the upper bound.

Proof.

- (i) In view of proposition 2.4, we have both lower and upper bound.
- (ii) Let S be a neighborhood set and $|V - S| = r$. Then there are at least q edges from $V - S$ to S and $|S| = p - r$, since $r \leq q$. we have $p - q \leq |S|$. Further, S is an η_m -set of G , which is an independent. Hence, the lower bound follows. Let S be a vertex cover at G with $|S| = \alpha_0(G)$ and $v \in V - S$. Then $S_1 = (S \cup v)$ is an η -set of G . Since there exist a vertex $u \in S$ adjacent to v and every vertex $w \in V - S_1$ is adjacent to atmost u , by Theorem 1.3, S is a maximal neighborhood set of G and hence upper bound follows.
- (iii) Let $S = (v_1, v_2, \dots, v_k)$ be a η_{pr} -set of G with matching $S' = (e_1, e_2, \dots, e_k)$, where each edge e_i connects two elements of S . Furthermore, let q be the number of edges in G having one vertex in S and the other in $V - S$. Since $\deg(v) \leq \Delta(G)$ for all $v \in S$ and each vertex in S has at least one neigh-

bor in S . $q \leq (\Delta(G) - 1)|S| = (\Delta(G) - 1)\eta_{pr}(G)$. Also, $q \geq |V - S| = p - \eta_{pr}(G)$. Hence, $|V - S| \leq \sum d(v_j) \leq (\Delta(G) - 1)|S|$. $p - \eta_{pr}(G) \leq (\Delta(G) - 1)\eta_{pr}(G)$ where $1 \leq j \leq k$ or $p \leq \Delta(G)\eta_{pr}(G)$. Thus, the lower bound follows. In a graph G with out isolated vertices, the vertices of any maximal independent set of edges form a paired-neighborhood set of G . Thus G has a paired-neighborhood neighborhood set if and only if the minimum vertex degree $\delta(G) \geq 1$. Hence, the upper bound follows.

- (iv) Let S be a η^{+2} - set of G . Then $q \leq |V - S| + |V - S|/2$. This proves the lower bound. If G is a kDN -graph with $\delta(G) \geq 2$, then any maximal independent set of edges form a $2DN$ -set of a graph G . Thus, the upper bound follows. □

Theorem 3.3 *For any graph G ,*

- (i) $\eta_m(G) = \eta(G)$ if and only if there exists an η -set S of G which is not independent, provided graph G with no triangles,
- (ii) $\eta_m(G) = \eta(G) + 1$ if and only if there exists an η -set S of G is independent, provided graph G with no triangles,
- (iii) $\eta_m(G) = p - q + 1$ if and only if each component of G is either a star or double star, where a double star is a tree with exactly two vertices of degrees greater than one.

Proof.

- (i) Suppose $\eta_m(G) = \eta(G)$ holds. On contrary, if every η -set S of G is independent, then $V - S$ is a vertex cover of G and hence it is an η -set of G , which is a contradiction. This proves the necessity. Suppose there exists an η -set S of G which is not independent, then there exist two vertices $u, v \in S$ such that u and v are adjacent and every vertex $w \in V - S$ is adjacent to at most one of u and v , since G has no triangles, thus by Theorem 1.2, S is an η_m -set of G and $\eta_m(G) = \eta(G)$ holds.
- (ii) Let S be a vertex cover at G with $|S| = \alpha_0(G)$ and $v \in V - S$. Then $S_1 = (S \cup v)$ is an η -set of G . Since there exist a vertex $u \in S$ adjacent to v and every vertex $w \in (V - S_1)$ is adjacent to at most u , by Theorem 1.3, S is an maximal neighborhood set of G and this implies to $\eta_m(G) \leq \alpha_0(G) + 1$. Hence by Proposition 2.4 and Theorem 1.2, $\eta_m(G) = \eta(G) + 1$ follows.
- (iii) Suppose the bound is attained. On contrary, suppose G contains a component which is neither a star nor a double star, then either $V - S$ is

not independent or S contains at least two edges. This implies that $\eta_m(G) > p - q + 1$, a contradiction. This proves the necessity. Sufficiency is easy to prove. $\square$

Let G_1 and G_2 be two graphs with no vertex in common. Then the join of G_1 and G_2 , denoted by $G_1 + G_2$, to be the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup F$, where $F = \{xy : x \in V(G_1), y \in V(G_2)\}$. Thus F consists of edges which join every vertex of a graph G_1 to every vertex of a graph G_2 . Further, if $G_1 = K_1$ and G_2 is a connected graph with $\Delta(G_2) = |V(G_2)| - 1$, then the join graph $G_1 + G_2 = H_1$, if G_1 is a complete graph K_t with $t \geq 2$ and G_2 be any graph, then the join graph $G_1 + G_2 = H_2$ and also if both G_1 and G_2 are connected graph with $\Delta(G_1) = |V(G_1)| - 1$ and $\Delta(G_2) = |V(G_2)| - 1$, then the join graph $G_1 + G_2 = H_3$. Let H be the collection of graphs H_1 , H_2 and H_3 .

Theorem 3.4 *Let G be a 2DN-graph with no isolated vertices. Then $\eta_{pr}(G) \leq \eta^{+2}(G)$. Further, the bound is attained if and only if G is isomorphic with K_p or mK_2 or H or $K_{t,t}$; $t \geq 1$.*

Proof. By the definition of $\eta_{pr}(G)$ and $\eta^{+2}(G)$. Clearly every 2DN-set is a paired neighborhood set of a graph G , then $\eta_{pr}(G) \leq \eta^{+2}(G)$ follows. Now, we prove the second part. Suppose $\eta_{pr}(G) = \eta^{+2}(G)$ holds. On contrary, suppose G is not isomorphic with K_p or mK_2 or H or $K_{t,t}$; $t \geq 1$. Then there exist at least two adjacent cut vertices u and v are adjacent to at least one end vertices, and thus u, v is a η_{pr} -set of G , but if G is a bipartite graph with no isolates, then $V(G)$ is a η^{+2} -set of G , this implies that $\eta_{pr}(G) < \eta^{+2}(G)$, which is a contradiction. Also, if $\deg(v) = \Delta$ which is not a complete graph, then there exist a vertex $u \in V$ such that u and v are adjacent, thus the set u, v is a η_{pr} -set of G and by Proposition 2.3 and Theorem 1.1, we have $\eta_{pr}(G) < \eta^{+2}(G)$, again a contradiction. This proves necessity, sufficiency is obvious. $\square$

Theorem 3.5 *If G has no isolated vertices and $\eta_{pr}(G) = 2\eta(G)$, then every η -set of G is an η_i -set of G .*

Proof. Let G be a graph having $\eta_{pr}(G) = 2\eta(G)$, and consider an η -set S . Suppose that S is not independent. Then, there is an adjacent pair of vertices in S , say u and v , we form a paired-neighborhood set for G by pairing u and v and pairing each vertex in $S - u, v$ with a neighbor in $V - S$. This is possible since the minimality of S implies that for each $x \in S$, either x has a private neighbor or x is isolated in $\langle S \rangle$. Let I be the set of isolates in S without private

neighbors. Now each vertex in I must have at least one neighbor in $V - S$, since G has no isolates. The minimality of S implies that no two vertices in I have a common neighbor. Hence, each vertex in $V - u, v$ can be paired with a neighbor forming a paired- neighborhood set of order $\eta(G) + \eta(G) - 2 < 2\eta(G)$, contrary to the hypothesis. Hence $\eta(G) = \eta_i(G)$ and since S was arbitrary, every minimum neighborhood set of G is independent. $\square$

Remark 3.6 The converse of the above theorem is not true. For example, consider a path P_6 , we have $\eta(P_6)=3$, $\eta_i(P_6)=3$ $\eta_{pr}(P_6)=4$.

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