Polarization of line radiation in the presence of external electric quadrupole and uniform magnetic fields

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Abstract

The polarization of emission lines formed in a medium immersed in external electric and magnetic fields is studied. The electric field is assumed to be quadrupolar in nature, while the magnetic field is uniform. We show that the quadrupole electric field produces line splitting which is characteristically different from the Zeeman effect. While the line components emitted along the quantization axis are circularly polarized in the Zeeman effect, they are, in contrast, linearly polarized in the case of a pure quadrupole electric field. The emission perpendicular to the quantization axis produces three linearly polarized components in Zeeman effect, whereas only two linearly polarized components are observed in the case of quadrupole electric fields. Lack of azimuthal symmetry in the quadrupole electric field leads to polarized line components which appear quite differently for different azimuthal angles of the line of sight.

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1. Introduction

In 1908, Hale [1] first observed circular polarization related to the Zeeman effect, in spectral lines originating in sunspots, which demonstrated the existence of magnetic fields in the solar atmosphere. In 1916, Wein [2] proposed the possibility of detecting electric fields through observations of linear polarization related to the Stark effect. But the electric fields originally suggested by Wein are ‘motional’ electric fields in the presence of a magnetic field. Charged particles which are in motion in a plasma generate [3,4] not only magnetic fields but also electric fields.

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As pointed out by Casini and Landi Degl’ Innocenti [5] “One of the most important diagnostic problems concerning the study of astrophysical plasma is represented by the detection of stationary electromagnetic fields”. To quote from [6], “Surprisingly little attention has been directed to the investigation of possible macroscopic electric fields in the solar atmosphere considering the important role that quasi stationary and wave related electric fields play in models of dynamic solar phenomena”. Considering the motional electric field \( \mathbf{E} = \frac{1}{c} (\mathbf{v} \times \mathbf{B}) \), where \( \mathbf{v} \) denotes the velocity of the atom and \( \mathbf{B} \) is the magnetic field in the sunspot, \( |\mathbf{E}| \) has been estimated to be of the order of \( 10 \text{ V cm}^{-1} \) [5] or even \( 10^3 \text{ V cm}^{-1} \) [7]. The practical difficulties in making observations at the solar limb were cited as a reason for neglecting the role of electric fields in solar plasma. The importance of electric fields in plasma physics has been recognized, following the pioneering work of Alfven, because of which modeling [8] of active phenomena in the solar atmosphere was taken up considering the presence of electric as well as magnetic fields. The recent work in the use of line polarization as a diagnostic of ambient electric fields in solar atmospheric features can be found in [9]. A rigorous formulation of the quantum problem of the hydrogen atom exposed to electric as well as magnetic fields was presented by Casini and Landi Degl’ Innocenti [5], widely generalizing the earlier work of Nguyen-Hoe et al. [10]. They gave expressions for the Stokes profiles emitted by a line transition using an earlier formalism [11], based on the density matrix concept. The choice of hydrogen lines was made because of the high relative abundance of hydrogen in the solar atmosphere and also the sensitivity of these lines to the linear Stark effect. Hydrogen line formation in plasmas has also been considered by Mathys [12] and Jaegle et al. [13]. Polarized line formation problems in the presence of electric and magnetic fields are discussed in [14]. A great deal of information on the radiation polarization is contained in the four Stokes parameters observed in the solar/stellar line spectra. Similarly, all the information on atomic polarization (that is created by, or is at the origin of the radiation polarization) is contained in the elements of the atomic density matrix as pointed out by Bommier [15].

When radiation is emitted by an atom making a transition from an upper level (with energy \( E_u \), angular momentum \( J_u \) and parity \( \pi_u \)) to a lower level (with energy \( E_l \), angular momentum \( J_l \) and parity \( \pi_l \)), the frequency \( \nu \) of the radiation, is given by \( h\nu = E_u - E_l \), which represents conservation of energy. The observed spectral lines in practice are, however, centered around this frequency with a Voigt profile generated by the convolution of Lorentzian and Gaussian line shapes. The angular distribution and polarization of the emitted radiation is governed by conservation of angular momentum and parity. For example, the Zeeman effect in the presence of magnetic field \( \mathbf{B} \) splits a \( J_u=1 \) level into a triplet of states with magnetic quantum numbers \( M=+1,0,-1 \) so that the radiations emitted from the \( M = \pm 1 \) states in a transition to \( J_l = 0 \) in the direction of \( \mathbf{B} \) are 100% circularly polarized. The Zeeman splitting polarizes the \( J_u = 1 \) atomic level, in the sense that the states with different \( M \) are unequally populated. “In the past only the finger prints of the Zeeman effect have been used, but more recently new and highly sensitive imaging polarimeters have given us access to other physical effects. In particular, a wealth of previously unknown spectral structures due to coherent scattering process have been uncovered. These phenomena show up in linear polarization as a new kind of spectrum (the so-called “second solar spectrum”), which bears little resemblance to the ordinary intensity spectrum” [16]. To interpret such high precision polarimetric data, non-magnetic resonance scattering polarization, Hanle effect (that is closely linked to the non-zero width of the excited state) [17,18], transitions from an upper level with \( J_u = 0 \) to a lower level with \( J_l = 1 \) which
is not a ground state (and hence is characterized by a non-zero width; lower level Hanle effect) \cite{20} and atomic level polarization due to hyperfine interactions \cite{19} have all been invoked, apart from the simple Zeeman effect \cite{21}. For a discussion on several mechanisms proposed to describe the second solar spectrum, see Stenflo \cite{22} and the edited volumes: Stenflo and Nagendra \cite{23}, Nagendra and Stenflo \cite{24}, Sigwarth et al. \cite{25}, Mathys et al. \cite{26} and Trujillo et al. \cite{27}.

The purpose of this paper is to present a theoretical investigation of the polarized spectral structure of the radiation emitted by an atom with upper level $J_u = 1$, when the atomic level is polarized due to an external electric field and the transition takes place to a lower ground level with $J_l = 0$. A comparison is made to the polarized spectral structure in the well-known case of atomic upper level polarization caused by the Zeeman effect. To make this investigation more realistic and perhaps also applicable in the astrophysical context, we consider as well, a simple scenario when the atomic level polarization arises due to the presence of the electric field together with an external magnetic field.

The plan of the paper is as follows: In Section 2, We consider the Coulomb interaction between an arbitrary external charge distribution and the atom. We express the interaction Hamiltonian as a sum of terms which fall off with distance $R$ as $R^{-(2l+1)}$, $l = 0, 1, \ldots, \infty$, where each term is a scalar product $(V_l \cdot Q_l)$. The $Q_{l,m}$ provide a comprehensive description of the charge distribution inside the atom, whereas the $V_{l,m}$ denote the multipole character of the external electric field. Since the total charge of the atom is zero and its dipole moment vanishes when the state of the atom is an eigen state of parity, the first two terms $l = 0$ and 1 do not contribute to the interaction Hamiltonian. We therefore consider the quadrupole interaction represented by $l = 2$ in this paper. In Section 3, the state of polarization of an atom with $J = 1$ when it is exposed to an external electric quadrupole field is discussed. This is compared with the well-known scenario of Zeeman effect in the presence of a uniform external magnetic field. The density matrix $\rho$ for the atom is expressed in terms of well known Fano statistical tensors and it is shown that the atomic state of polarization could be non-oriented when the asymmetry parameter $\eta$ of external electric quadrupole field is non-zero. The state of polarization of an atom when the electric field is supplemented by an additional external magnetic field is also discussed for a particularly simple orientation of the magnetic field. More general relative orientations of the magnetic field vis-a-vis the electric quadrupole field will be taken up for a detailed study in a sequel to this paper. In Section 4, the polarization of radiation emitted by a polarized atom exposed to an external electric quadrupole field is studied. We use the density matrix formalism for radiation, and derive explicit expressions for the Stokes parameters $I, Q, U, V$ for each of the spectral lines emitted in an arbitrary direction $(\theta_k, \phi_k)$. Section 5 is devoted to a detailed discussion of the theoretical results obtained in this paper. For this purpose, explicit formulae are derived for the Stokes line profiles in all the three cases viz., (i) pure magnetic field, (ii) pure electric quadrupole field and (iii) the combined electric and magnetic fields. Numerical results are presented for some selected directions of emission $(\theta_k, \phi_k)$. In the case of (iii) the Stokes line profiles are also presented as functions of the relative strengths of the electric and magnetic fields.

### 2. Interaction of atom with an arbitrary external electric field

Consider an atom located at the origin, O (see Fig. 1) and surrounded by an arbitrary charge distribution $\Phi(R)$, generating the electric field to which the atom is exposed. The atom itself consists of a nucleus carrying an electric charge $Ze$. The spatial distribution of the electric charge $Ze$ within the
Fig. 1. Atom exposed to an arbitrary charge distribution. See Section 2 for details.

The nucleus may effectively be defined by a wave function \( \Psi_{\text{nuc}} \), which may, in general, be expressed as a linear combination of \( \Psi_{I}(r_0) \) with nuclear spin \( I \) and its projection \( v \) along the axis of quantization or the \( Z \)-axis. The \( Z \) electrons surrounding the nucleus may be characterized by the atomic wave function \( \Psi_{\text{atom}}(r_1, \ldots, r_z) \), which may be expressed, in general, in terms of \( \Psi_{J,M}(r_1, \ldots, r_z) \) with spin \( J \) and its projection \( M \) along the axis of quantization. The Coulomb interaction between the external charge distribution and the atom is given by the interaction Hamiltonian

\[
H_{\text{int}} = \int d^3 \Phi(R) \left[ \frac{Ze}{|R - r_0|} - \sum_{k=1}^{z} \frac{e}{|R - r_k|} \right],
\]

where \( r_1, \ldots, r_k, \ldots, r_z \) denote the locations of the electrons. Expressing

\[
\frac{1}{|R - r_k|} = \frac{1}{R[1 + (r_k/R)^2 - 2(r_k/R) \cos \theta_k]^{1/2}}, \quad k = 0, 1, 2, \ldots, Z,
\]

where \( r_k \cdot R = r_k R \cos \theta_k \), we note that the generating function for Legendre polynomials

\[
\frac{1}{(1 + x^2 - 2x \cos \theta)^{1/2}} = \sum_{l=0}^{\infty} x^l P_l(\cos \theta), \quad x < 1,
\]

enables us to express Eq. (2) in terms of the Legendre polynomials \( P_l(\cos \theta_k) \), which in turn may be expressed as

\[
P_l(\cos \theta_k) = \frac{4\pi}{2l + 1} \sum_{m=-l}^{l} (-1)^m Y_{l,m}(\hat{r}_k) Y_{l,-m}(\hat{R}),
\]
in terms of spherical harmonics $Y_{l,m}$. We may then introduce irreducible tensors $C_{l,m}(\mathbf{r})$ of rank $l$ through

$$C_{l,m}(\mathbf{r}) = \sqrt{\frac{4\pi}{2l+1}} r^l Y_{l,m}(\hat{\mathbf{r}}),$$

which are homogeneous polynomials of degree $l$ in terms of the components $x, y, z$ of $\mathbf{r}$ and express $H_{\text{int}}$ in the elegant form

$$H_{\text{int}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \langle V_l \cdot Q_l \rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-1)^m V_{l,m} Q_{l,m}.$$  

Clearly, each term $(V_l \cdot Q_l)$ is a scalar product of two irreducible tensors of the same rank, $l$ where

$$V_{l,m} = \int d^3R \Phi(\mathbf{R}) C_{l,-m}(\mathbf{R})/R^{2l+1}$$

are determined entirely by the external charge distribution, while

$$Q_{l,m} = ZeC_{l,m}(\mathbf{r}_0) - e \sum_{k=1}^{Z} C_{l,m}(\mathbf{r}_k) = Q_{l,m}(\text{nuc}) + Q_{l,m}(\text{atom})$$

provide a comprehensive description of the charge distribution within the atom. The first term $Q_{l,m}(\text{nuc})$ in Eq. (8) refers to the charge distribution in the nucleus, while the second part $Q_{l,m}(\text{atom})$ is a sum of $Z$ terms dependent on the spatial distribution of the electrons in the atom. We note that the $l=0$ term in Eq. (6) is a product of $V_{0,0}$ and $Q_{0,0}$, given by

$$V_{0,0} = \int d^3R \Phi(\mathbf{R})/R, \quad Q_{0,0} = Ze - \sum_{k=1}^{Z} e = 0$$

and thus it represents the Coulomb interaction between the external charge distribution $\Phi(\mathbf{R})$ and the atom, considered as a point object with total charge $Q_{0,0}$ located at the origin ‘O’. Since $Q_{0,0} = 0$ for a neutral atom, this term contributes zero to $H_{\text{int}}$. The higher order contributions, $l > 0$ to Eq. (6) may be referred to as the $2^l$ multipole interaction terms and they arise due to the distributed nature of the electric charge in the atom, although the atom is neutral as a whole. Moreover, one has to take the expectation values $\langle \Psi_{\text{nuc}}^\dagger|Q_{l,m}(\text{nuc})|\Psi_{\text{nuc}}\rangle$ and $\langle \Psi_{\text{atom}}^\dagger|Q_{l,m}(\text{atom})|\Psi_{\text{atom}}\rangle$ to estimate these contributions at the atomic level. For this, we note that $C_{l,m}(\mathbf{r}_k), k=0,1,2,\ldots,Z$ are irreducible tensor operators of rank $l$ and as such the matrix elements of these operators may be evaluated using the Wigner–Eckart theorem as

$$\langle \Psi_{l',v'}^\dagger|Q_{l,m}(\text{nuc})|\Psi_{l,v}\rangle = C(I, l, l'; v, m, v')(I'|Q_{l}(\text{nuc})||I),$$  

$$\langle \Psi_{j',M'}^\dagger|Q_{l,m}(\text{atom})|\Psi_{j,M}\rangle = C(J, l, J'; M, m, M')(J'|Q_{l}(\text{atom})||J),$$

employing the same notations as in [28]. The matrix elements in Eq. (10) with $I = I' = v = v'$ for the nucleus and Eq. (11) with $J = J' = M = M'$ for the atomic electrons may be referred to as the electric $2^l$-pole moments $Q_{l}(\text{nuc})$ for the nucleus and $Q_{l}(\text{atom})$ for the atom respectively. The electric dipole moments with $l = 1$ vanish, since the nuclear and the atomic wave functions are
usually good eigen states of parity. Since \( \langle \Psi_{\text{atom}} | Q_{l,m(\text{atom})} | \Psi_{\text{atom}} \rangle = 0 \) and \( \langle \Psi_{\text{nuc}} | Q_{l,m(\text{nuc})} | \Psi_{\text{nuc}} \rangle = 0 \) for \( l = 1 \), there is no contribution to the expectation value of Eq. (6) from \( l = 1 \) although the \( l = 1 \) term in Eq. (6) can lead to transitions between atomic states with opposite parity, when \( \Psi_{\text{atom}} \) is a superposition of states with different parities as in the Stark effect. Even when \( \Psi_{\text{atom}} \) is a good eigen state of parity, second-order perturbation theory can lead to non-zero expectation values with the \( l = 1 \) terms. But, it should be noted that second order Stark effect is of order \( e^2 \) rather than \( e \). Thus, the lowest order non-zero contributions, which are of order \( e^2 \), to Eq. (6) come from \( l = 2 \) viz., the electric quadrupole interactions, provided \( I \geq 1 \) and \( J \geq 1 \) for the nuclear and atomic spins, as otherwise the respective quadrupole moments proportional to \( Q_2(\text{nuc}) = \langle \Psi_{l,I} | Q_{2,0(\text{nuc})} | \Psi_{l,I} \rangle \) and \( Q_2(\text{atom}) = \langle \Psi_{l,J,I} | Q_{2,0(\text{atom})} | \Psi_{l,J,I} \rangle \) vanish, due to the vanishing of the Clebsch–Gordan coefficients in Eqs. (10) and (11) in such a case. Since \( Q_2(\text{nuc}) \ll Q_2(\text{atom}) \), we neglect the nuclear contribution and represent the \( l = 2 \) term in \( H_{\text{int}} \) as a \((2J+1) \times (2J+1)\) matrix in atomic spin space with elements given by

\[
\langle JM | H_{\text{int}}(l = 2) | JM' \rangle = Q \left( \frac{(J+1)(2J+3)}{J(2J-1)} \right)^{1/2} \sum_{m=-2}^{2} (-1)^m V_{2,-m} C(J, 2, J', m, m, M),
\]

where \( Q = Q_2(\text{atom}) = C(J, 2, J; J, 0, J) \langle J \| Q_2(\text{atom}) \| J \rangle \). It is interesting to note that the elements \( V_{2,m} \) of the second rank irreducible tensor could always be expressed in terms of the elements of a traceless symmetric second rank Cartesian tensor, \( V_{\alpha\beta} = V_{\beta\alpha}, \ \alpha, \beta = x, y, z \) through

\[
V_{2,0} = \frac{1}{4} V_{zz}, \quad V_{2,\pm 1} = \mp \frac{1}{2\sqrt{6}} (V_{xz} \pm iV_{yz}), \\
V_{2,\pm 2} = \frac{1}{4\sqrt{6}} (V_{xx} - V_{yy} \pm 2iV_{xy})
\]

(13) and vice versa. It is also interesting to note that such a Cartesian tensor, \( V_{\alpha\beta} \) with \( V_{xx} + V_{yy} + V_{zz} = 0 \) could be brought to the diagonal form through a rotation of the Cartesian coordinate system. We call the coordinate system where the Cartesian tensor \( V_{\alpha\beta} \) is diagonal such that \( |V_{zz}| \gg |V_{xx}| \gg |V_{yy}| \) as the principle axes frame (PAF). We note from Eq. (13) that \( V_{2,\pm 1} = 0 \) and \( V_{2,\pm 2} = V_{2,\mp 2} \) in PAF. Thus Eq. (12) assumes the form

\[
\langle JM | H_{\text{int}}(l = 2) | JM' \rangle = A \left[ \delta_{MM'} \left\{ 3M^2 - J(J+1) \right\} + \frac{\eta}{2} \delta_{M,M'+2} \\
\times \left\{ (J + M - 1)(J + M)(J - M + 1)(J - M + 2) \right\}^{1/2} + \frac{\eta}{2} \delta_{M,M'-2} \left\{ (J + M + 1)(J - M)(J - M - 1)(J + M + 2) \right\}^{1/2} \right],
\]

(14)

where the strength \( A \) and the asymmetry parameter \( \eta \) respectively, are given by

\[
A = \frac{Q V_{zz}}{4J(2J-1)}, \quad \eta = \frac{V_{xx} - V_{yy}}{V_{zz}}.
\]

(15)
It may be noted that the contributions of terms of higher order in $l$ to Eq. (6) get progressively weaker, since $V_{l,m}$ given by Eq. (7) varies with $R$ as $1/(R^{2l+1})$. It may also be noted that $l$ must be even due to atomic states being parity eigen states to ensure $\langle J || Q_l (atom) || J \rangle \neq 0$, for a $2^l$-pole interaction to exist between the atom and the external electric field. Thus, atomic states with $J = 2$ could be expected to be susceptible to the electric hexadecapole fields under favorable conditions. We restrict ourselves here to the polarization of $J = 1$ atomic states interacting with electric quadrupole components generated by an external electric charge distribution.

3. Atomic polarization in external electric and magnetic fields

The Hamiltonian $H_{\text{int}}$ for an atom interacting with an external magnetic field $B$ and an external electric quadrupole field represented by the electric field gradient tensor $V_{\alpha \beta}$, $\alpha, \beta = x, y, z$ has a well-known \[29,30\] from

$$H_{\text{int}} = -g J \cdot B + A[\{2 J_x^2 - J_y^2 - J_z^2\} + \eta\{J_x^2 - J_y^2\}],$$

where $g$ denotes the gyromagnetic ratio and $J_x, J_y, J_z$ denote components of the spin operator $J$ with respect to PAF, defined in Section 2, and $J(J + 1)$ denotes the eigenvalue of $J \cdot J$ in natural units. As the wave functions $\Psi_{J,M}$ of the atom with magnetic quantum number $M = J, J - 1, \ldots, -J$ are eigen states with same parity, the electric dipole moment vanishes, and hence the interaction with an external electric dipole field does not appear in Eq. (16).

3.1. Case of atom in a pure magnetic field

If $A = 0$ and if $B$ defines the $Z$-axis or the axis of quantization, the eigen states of $H_{\text{int}}$ are readily seen to be $\Psi_{J,M}$ with corresponding eigen values $E_M = -gBM$ where $E = 0$ corresponds to energy $E_u$ of the unsplit level when $A = 0, B = 0$. This leads to the well-known Zeeman splitting with equal spacing between the $(2J + 1)$ levels, such that $M = +J$ level is the lower most and $M = -J$ level is the upper most. This is shown in Fig. 2b for the case $J = 1$.

3.2. Case of atom in a pure electric quadrupole field

If $A \neq 0$ but $B = 0$, the energy eigen values and the corresponding eigen states of $H_{\text{int}}$ are determined by diagonalizing the matrix form of $H_{\text{int}}$ with respect to basis states $\Psi_{J,M}$ defined with the $Z$-axis of PAF chosen as axis of quantization. In the particular case of $J = 1$, the matrix $H_{\text{int}}$ is given explicitly by

$$H_{\text{int}} = \begin{pmatrix} A & 0 & \eta A \\ 0 & -2A & 0 \\ \eta A & 0 & A \end{pmatrix}$$
Fig. 2. Comparison of level splitting for a $J = 1$ state in pure magnetic and pure electric quadrupole fields. See Sections 3, 5.1 and 5.2 for details. (a) $A = 0, B = 0$; (b) $A = 0, B \neq 0$; (c) $B = 0, A \neq 0, \eta = 1$; (d) $B = 0, A \neq 0, 0 < \eta < 1$; (e) $B = 0, A \neq 0, \eta = 0$.

and the energy eigen values are readily found to be

$$E_x = A(1 - \eta), \quad E_y = A(1 + \eta), \quad E_z = -2A$$

with the corresponding eigen states given by

$$\Psi_x = 2^{-1/2}(\Psi_{1,-1} - \Psi_{1,1}), \quad \Psi_y = 2^{-1/2}(\Psi_{1,-1} + \Psi_{1,1}), \quad \Psi_z = \Psi_{1,0},$$

which are orthonormal and satisfy the eigen value equations

$$J_\alpha \Psi_\alpha = 0, \quad \alpha = x, y, z.$$

The level scheme for this case is shown in Fig. 2.

3.2.1. Dependence of level splitting on the asymmetry parameter $\eta$

In the particular case, where the asymmetry parameter $\eta$ of the field is zero, the states $\Psi_x, \Psi_y$ are degenerate with $E_x = E_y = A$ (see Fig. 2e). In such a situation, any orthonormal pair of states which are linear superpositions of $\Psi_x, \Psi_y$ are also eligible to be considered as eigen states of energy $A$. Thus in particular, $\Psi_{1,+1}$ and $\Psi_{1,-1}$ are eigen states with energy $A$ so that $\Psi_{J,M}$ could themselves be considered as eigen states of $H_{\text{int}}$ with energy eigen values $E_M$ given by $E_{+1} = E_{-1} = A$, and $E_0 = -2A$.

On the other hand, if $\eta = 1$, then $E_x = 0$ (see Fig. 2e). Further $E_y$ and $E_z$ take values $2A$ and $-2A$, respectively, so that the energy levels are equally spaced, similar to the case of Zeeman effect. But $\Psi_x$ is the middle level in contrast to $\Psi_z = \Psi_{1,0}$ being the middle one in the case of Zeeman effect. For $0 < \eta < 1$, the spacings between the levels are unequal with no level at $E = 0$, which corresponds to $E_u$ when $A = B = 0$ (see Fig. 2d).
3.3. The general case of level splitting when $A \neq 0$, $B \neq 0$

When both the electric and magnetic fields are present, one may either (i) choose $B$ to define the axis of quantization or (ii) choose the $Z$-axis of the PAF of the electric field gradient tensor $V_{\alpha\beta}$ to be the axis of quantization. It is more convenient to choose the latter and define $-gB = D$, in terms of its three components $D_x, D_y, D_z$ in PAF. The problem of determining the energy eigen values $E_1, E_2, E_3$ along with the corresponding eigen states $\Psi_1, \Psi_2, \Psi_3$ becomes easier if $B$ is along one of the principal axes, say, the $Z$-axis of PAF, i.e. $D = (0,0,D)$. The matrix form of the Hamiltonian will then become

$$H_{\text{int}} = \begin{pmatrix} A + D & 0 & \eta A \\ 0 & -2A & 0 \\ \eta A & 0 & A - D \end{pmatrix},$$

instead of Eq. (17). The energy eigen values are easily found to be

$$E_1 = -2A, \quad E_{2,3} = A \pm \sqrt{A^2\eta^2 + D^2}$$

and the corresponding eigen states are given by

$$\Psi_1 = \Psi_{1,0}, \quad \Psi_2 = a\Psi_{1,-1} + b\Psi_{1,1}, \quad \Psi_3 = b\Psi_{1,-1} - a\Psi_{1,1},$$

where the coefficients $a$ and $b$ are real and are given by

$$a = \frac{A\eta - D + (A^2\eta^2 + D^2)^{1/2}}{2[A^2\eta^2 + D^2 + A\eta(A^2\eta^2 + D^2)^{1/2}]^{1/2}},$$
$$b = \frac{A\eta + D + (A^2\eta^2 + D^2)^{1/2}}{2[A^2\eta^2 + D^2 + A\eta(A^2\eta^2 + D^2)^{1/2}]^{1/2}}.$$

The level scheme for this case is shown in Fig. 3. Note that the equal spacing of energy levels can occur here only when $(A^2\eta^2 + D^2)^{1/2} = A$. It may be noted from Fig. 2 that the energy levels $\Psi_x, \Psi_y$ occur with energy values $E_x, E_y$ are above $E = 0$ and the eigen value $E_x = 0$ only in the limiting case of $\eta = 1$. In contrast, we find now that the states $\Psi_2, \Psi_3$ could both be above $E = 0$ or they may also lie on opposite sides of $E = 0$. The problem becomes more complicated if $B$ is along any arbitrary direction with respect to PAF. The general case of arbitrary orientation of $B$ will be discussed in a forthcoming paper.

3.4. Density matrix $\rho$ for the assembly of atoms

In general, therefore, it follows that if the energy eigen values $E_i$ and the corresponding eigen states $\Psi_i$, $i = 1, 2, 3$ for $H_{\text{int}}$ are known, the populations in $\Psi_i$ are given by the probabilities

$$p_i = P^{-1}e^{-E_i/k_BT}, \quad P = \sum_i e^{-E_i/k_BT},$$

where $k_B$ denotes the Boltzmann constant, $T$, the absolute temperature and $P$, the partition function.
Fig. 3. Level splitting for \( J = 1 \) state in combined external magnetic and electric quadrupole fields, \( A \neq 0, D \neq 0, \) and \( 0 \leq \eta \leq 1. \) See Sections 3 and 5.3 for details. (a) \( A = 0, D = 0; \) (b) \( A \neq 0, D \neq 0, (A^2\eta^2 + D^2)^{1/2} < A; \) (c) \( A \neq 0, D \neq 0, (A^2\eta^2 + D^2)^{1/2} > A; \) (d) \( A \neq 0, D \neq 0, (A^2\eta^2 + D^2)^{1/2} = A. \)

The density matrix for a statistical assembly of atoms is then given by

\[
\rho^o = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix},
\]

if the rows and columns are labeled by the eigen states \( \Psi_1, \Psi_2, \Psi_3. \) Since the states \( \Psi_i \) are known in terms of \( \Psi_{1,M}, \) using Eq. (23), it is clear that we may express

\[
\Psi_i = \sum_M c_M^i \Psi_{1,M}, \quad \sum_M |c_M^i|^2 = 1,
\]

where the complex expansion coefficients are \( c_M^i = \langle \Psi_{1,M} | \Psi_i \rangle \) for \( i = 1, 2, 3 \) and \( |c_M^i|^2 = c_M^{i*} c_M^i. \) Using Eq. (27), the density matrix \( \rho \) with respect to the states \( \Psi_{1,M} \) for the atomic system is readily obtained in terms of its elements

\[
\rho_{MM'} = \sum_i c_M^i p_i c_{M'}^{i*} = \frac{1}{3} \sum_{k=0}^2 (-1)^{q} [k] \eta C(1, k, 1; M', -q, M),
\]

where \( \eta = A^2\eta^2 + D^2 \) and \( C(1, k, 1; M', -q, M) \) are the Clebsch-Gordan coefficients.
where \( t^k_q \) denotes Fano statistical tensors [31] of rank \( k \), \( k = \sqrt{2k + 1} \) and the notation for Clebsch–Gordan coefficient follows Rose [28]. Explicitly, we have

\[
\rho = \frac{1}{3} \begin{bmatrix}
1 + \sqrt{3/2}t^1_0 + \sqrt{1/2}t^2_0 & \sqrt{3/2}(t^1_{-1} + t^2_{-1}) & \sqrt{3}t^2_{-2} \\
-\sqrt{3/2}(t^1_1 + t^2_1) & 1 - \sqrt{2}t^2_0 & \sqrt{3/2}(t^1_{-1} - t^2_{-1}) \\
\sqrt{3}t^2_{2} & -\sqrt{3/2}(t^1_1 - t^2_1) & 1 - \sqrt{3/2}t^2_0 + \sqrt{1/2}t^2_0
\end{bmatrix},
\]

(29)

where the rows and columns are labeled by states \( \Psi_{1,M}, M = +1, 0, -1 \) and the Fano [31] statistical tensors \( t^k_q \) denote the average expectation values

\[
t^k_q = \text{Tr}(\rho t^k_q),
\]

(30)
of irreducible tensor operators \( \tau^k_q \) of rank \( k \), whose normalization factors follow the Madison convention [32]. The \( \tau^k_q \) are given explicitly by

\[
\begin{align*}
\tau^0_0 &= 1, \\
\tau^1_0 &= (\sqrt{3/2})J_z, \\
\tau^1_{\pm 1} &= \mp(\sqrt{3/2})(J_x \pm iJ_y), \\
\tau^2_0 &= (1/\sqrt{2})(2J^2_z - J^2_x - J^2_y), \\
\tau^2_{\pm 1} &= \mp(\sqrt{3/2})[(J_x \pm iJ_y)J_z + J_z(J_x \pm iJ_y)], \\
\tau^2_{\pm 2} &= (\sqrt{3/2})(J_x \pm iJ_y)^2,
\end{align*}
\]

(31)
in terms of the spin operators \( J_x, J_y, J_z \). The \( t^k_q \) and their complex conjugates \( t^k_q^* \) are related through

\[
t^k_q^* = (-1)^q t^k_{-q},
\]

(32)

and they transform under rotations from a coordinate system I to a coordinate system II, according to

\[
(t^k_q)_\Pi = \sum_{q' = -k}^k D^k_{q'q}(\alpha, \beta, \gamma)(t^k_{q'})_\Pi,
\]

(33)

where \( D^k \) denote standard rotation matrices and \( (\alpha, \beta, \gamma) \), the Euler angles as defined by Rose [28]. The \( t^1_q \) are related to the components \( P_x, P_y, P_z \) of an axial vector \( P \) through

\[
\begin{align*}
t^0_0 = (\sqrt{3/2})P_z, \\
t^1_0 = \mp(\sqrt{3/2})(P_x \pm iP_y),
\end{align*}
\]

(34)

while \( t^2_q \) are expressible as

\[
\begin{align*}
t^2_0 = P_{zz}/\sqrt{2}, \\
t^2_{\pm 1} = \mp(P_{xz} \pm iP_{yz})/\sqrt{3}, \\
t^2_{\pm 2} = (P_{xx} - P_{yy} \pm 2iP_{xy})/2\sqrt{3},
\end{align*}
\]

(35)
in terms of a second rank Cartesian tensor \( P_{zz} \), which is symmetric and traceless. As such there exists a Cartesian coordinate system, which is referred to [33,34] as the principal axes of alignment
frame (PAAF) where $P_{x\beta} = \delta_{x\beta}P_{xx}$ such that $P_{xx} + P_{yy} + P_{zz} = 0$. Note that the alignment of the system is defined by $P_{x\beta}$. Note also that Eq. (35) is similar to Eq. (13) of Section 2, except for an overall factor $(2\sqrt{2})$. The normalization factors in Eq. (13) were chosen to obtain the form for $H_{int}$ in agreement with well-known form in the context of NMR and NQR [29]. On the other hand, the normalization here follows the Madison convention [32] for $g_{FS_k} q$. It is clear from Eq. (35) that $t_{\pm 1}^2 = 0$, $t_2^2 = t_{-2}^2$ in PAAF.

It has been shown [35] that $t_q^2$ are expressible as

$$t_q^2 = t(\hat{Q}(1) \otimes \hat{Q}(2))_q^2 = t \sum_{q_1} C(1,1,2; q_1, q_2, q) \hat{Q}_{q_1}(1) \hat{Q}_{q_2}(2), \quad (36)$$

in terms of a strength parameter $t \geq 0$ for tensor polarization and two unit axial vectors $\hat{Q}(1)$ and $\hat{Q}(2)$, while $P = PP$ in terms of the strength $P = |P|$ for vector polarization and a unit axial vector $\hat{P}$. Thus, the state of polarization of the spin 1 atomic system is, in general, characterized by the strengths $P$ and $t$ of vector and tensor polarizations respectively and one axis $\hat{P}$ for vector polarization and two axes $\hat{Q}(1)$ and $\hat{Q}(2)$ for tensor polarization. In general, the three axes $\hat{P}$, $\hat{Q}(1)$ and $\hat{Q}(2)$ could be arbitrary. In particular when they become collinear and define a common axis, viz., the ‘axis of orientation’, the system is said to be ‘oriented’ [36,37] with a manifestation of cylindrical symmetry with respect to the ‘axis of orientation’.

Clearly the system is ‘oriented’ in the particular case 3.1, with the ‘axis of orientation’ defined by the magnetic field. In the particular case when $\eta = 0$, it is ‘aligned’ as well as ‘oriented’ as discussed in the case 3.2. The axes $\hat{Q}(1)$ and $\hat{Q}(2)$ are distinct in the case 3.2 if $\eta \neq 0$, where the system is ‘aligned’ with no cylindrical symmetry with respect to the $Z$ axis of PAAF. The density matrix of Eq. (29) in this case 3.2, with no external magnetic field, assumes the simple form

$$\rho = \frac{1}{3} \begin{bmatrix} 1 + \frac{1}{\sqrt{2}} t_0^2 & 0 & \sqrt{3} t_2^2 \\ 0 & 1 - \sqrt{2} t_0^2 & 0 \\ \sqrt{3} t_2^2 & 0 & 1 + \frac{1}{\sqrt{2}} t_0^2 \end{bmatrix}, \quad (37)$$

in PAAF, where $t_2^2 = t_{-2}^2$. Clearly the eigen states of $\rho$ viz; $\Psi_1$, $\Psi_3$, $\Psi_2$ are identifiable as $\Psi_x$, $\Psi_y$, $\Psi_z$ with respect to PAAF. The respective eigen values $p_1$, $p_3$, $p_2$ correspond to $3^{-1}(1 + t_0^2/\sqrt{2} \pm \sqrt{3} t_2^2)$ and $(1 - \sqrt{2} t_0^2)$ in PAAF.

It is worth noting that a pure magnetic field produces vector as well as tensor polarization, whereas a pure electric quadrupole field produces only tensor polarization. Thus, vector polarization of the spin-1 system may be considered as a signature for the presence of a magnetic field.

On the other hand, a pure magnetic field can generate only an oriented spin-1 system with axis of orientation along the direction of the magnetic field, whereas either a pure electric quadrupole field with $\eta \neq 0$ or an electric quadrupole field together with a magnetic field in an arbitrary direction generates a ‘non-oriented’ [38,39] spin-1 system. Thus, the ‘non-oriented’ polarization of a spin-1 system provides a signature for the presence of a quadrupole electric field.

Since a pure electric quadrupole field with $\eta = 0$ produces an oriented spin-1 system, it should also be borne in mind that an oriented spin-1 system does not rule out the presence of an electric
quadrupole field. It is also interesting to note that the Fano statistical tensors $t^k_q$ for an ‘oriented’ system satisfy $n^2 - n - 2$ constraints [36] where $n = 2J + 1$. This may be visualized by noting that a spin $J$ system, in general, has $n^2 - 1$ degrees of freedom, when $Tr \rho = 1$. They may be identified with a set of $J(2J + 1)$ independent axes and $2J$ independent scalars [35]. In the case of an oriented system, all the axes collapse into one so that the number of degrees of freedom get reduced to $n + 1$. Therefore, any spin system with $J \geq 1$ can exhibit non-oriented states, whereas for $J = 1/2$, the system is always oriented. In the case of pure electric field, the aligned spin-1 system is characterized by two independent axes and one scalar if $\eta \neq 1$. If $\eta = 1$, the two axes collapse into one and the aligned system exhibits cylindrical symmetry as well.

4. Polarization of emitted line radiation

4.1. States of photon polarization

It is well-known that electromagnetic radiation is transverse with electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ vibrating perpendicular to the direction $\mathbf{k}(\theta_k, \phi_k)$ of propagation. It is also well-known that $\mathbf{E}$ and $\mathbf{H}$ may be expressed in terms of the vector potential $\mathbf{A}$ through

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{H} = \nabla \times \mathbf{A},$$

so that a plane wave propagating along $\mathbf{k}$ is represented by

$$\mathbf{A} = A\tilde{\mathbf{e}} \exp(-i\omega t) \exp(i\mathbf{k} \cdot \mathbf{r}),$$

where $\tilde{\mathbf{e}}$ denotes the polarization. The condition $\tilde{\mathbf{e}} \cdot \mathbf{k} = 0$ ensures transversality. The amplitude $A = |\mathbf{A}|$ and $k = |\mathbf{k}| = 2\pi\nu = \omega$ in natural units, where $\nu$ denotes the frequency. In a frame of reference $\mathcal{F}_0$ with its $Z$-axis along $\mathbf{k}$, the polarization vector $\tilde{\mathbf{e}}$ may therefore be expressed as a linear combination of unit vectors $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ along the $X$ and $Y$ directions. This frame $\mathcal{F}_0$ can be obtained through a rotation $R(\phi_k, \theta_k, -\phi_k)$ from the lab-frame $\mathcal{F}_L$, where $\mathbf{k}$ has polar coordinates $(k, \theta_k, \phi_k)$. Polarized plane waves as in Eq. (39) correspond to eigen states of linear momentum with eigen values $k$. If $|\tilde{\mathbf{e}}_1\rangle$ and $|\tilde{\mathbf{e}}_2\rangle$ denote a pair of orthonormal linear states of polarization, such that

$$|\tilde{\mathbf{e}}_1\rangle = |\tilde{\mathbf{e}}_x\rangle \cos \alpha + |\tilde{\mathbf{e}}_y\rangle \sin \alpha,$$

$$|\tilde{\mathbf{e}}_2\rangle = -|\tilde{\mathbf{e}}_x\rangle \sin \alpha + |\tilde{\mathbf{e}}_y\rangle \cos \alpha,$$

any arbitrary state $|\tilde{\mathbf{e}}\rangle$ of polarization may be defined through

$$|\tilde{\mathbf{e}}(\alpha\beta)\rangle = |\tilde{\mathbf{e}}_1\rangle \cos \beta + i|\tilde{\mathbf{e}}_2\rangle \sin \beta,$$

where $0 \leq \alpha < \pi$ and $-\pi/4 \leq \beta \leq \pi/4$. The above state of polarization defined by Eq. (41) may then be represented uniquely by a point with polar coordinates $\theta = (\pi/2) + 2\beta$, $\phi = 2\alpha$ on the Poincaré sphere, where the equatorial belt defines all possible linear states of polarization (plane polarized states), the upper and lower hemispheres represent respectively all possible right and left elliptical polarized states, with the north and south poles corresponding to right and left circular
polarized states. Note that any two diametrically opposite points on the Poincaré sphere correspond to mutually orthogonal states viz., $|\hat{e}(\pi, \beta)\rangle$ and $|\hat{e}((\pi/2) + \pi, -\beta)\rangle$. The left and right circular states of polarization are thus defined through

$$|\mu = \pm 1\rangle = \frac{1}{\sqrt{2}}(|\hat{e}_1\rangle + i\mu|\hat{e}_2\rangle),$$

which correspond respectively to $\beta = \pm \pi/4$. Polarized plane wave states given by Eq. (39), and properly normalized, may then be expressed using the well-known multipole expansion [28]

$$|k, \mu\rangle = (2\pi)^{1/2} \sum_{L=1}^{\infty} \sum_{M=-L}^{L} i^L (2L + 1)^{1/2} D_{M\mu}^{L}(\phi_k, \theta_k, -\phi_k)$$

$$\langle LM(magnetic) + i\mu|LM(electric)\rangle),$$

in terms of the ‘magnetic’ and ‘electric’ states of opposite parity with total angular momentum $L$ and its projection $M$ along the $Z$-axis.

4.2. Polarization density matrix for radiation

When an atom makes a transition from an upper level with energy $E_u$, spin-1 and parity $\pi_u$ to a lower level with energy $E_l$, spin-0 and parity $\pi_l$, angular momentum and parity are conserved. The transition matrix elements may then be expressed using Eq. (43) in the form

$$(k, \mu|T|1, M) = D_{M\mu}^{1}(\phi_k, \theta_k, -\phi_k)^*,$$

where the transition strength $\mathcal{J} = -i\sqrt{3}(2\pi)^{1/2}\langle 0||T||1\rangle$ may be displayed in the form

$$\mathcal{J} = \frac{1}{2}[\mathcal{H}\{\pi_u\pi_l + 1\} + \delta\{\pi_u\pi_l - 1\}],$$

indicating that the $\mathcal{J}$ is of either a magnetic or an electric dipole type, depending on the parities of the upper and lower levels. If the lower level is the ground state and if the transition is from an upper state $\Psi_i$ which is of the form given by Eq. (27), and $\Psi_i$ has a width $\Gamma_i$, then the transition is described by matrix elements denoted by

$$T_{\mu,i} = \mathcal{J} f(i) \sum_{M=1}^{1} c_{M\mu}^{1}(\phi_k, \theta_k, -\phi_k)^*,$$

where

$$f(i) = \frac{1}{(E_i - E_l - \omega) - i\Gamma_i}$$

incorporates the frequency dependence due to the natural width $\Gamma_i$. The density matrix $\rho^\nu$ describing the state of polarization of radiation can be written as

$$\rho^\nu_{\mu\mu'} = \sum_{i=1}^{3} T_{\mu,i} T_{\mu',i}^* = \sum_{i=1}^{3} \rho^\nu_{\mu\mu'}(i),$$

(48)
where \( \mu, \mu' \) take values \( \pm 1 \), the probabilities \( p_i \) are in general given by Eq. (25) in Section 3 and \( \rho^\gamma(i) \) correspond to the transition from an upper level \( \Psi_i \) to the lower level \( |0,0\rangle \) representing the ground state. The elements of the density matrices \( \rho^\gamma(i) \) are the form

\[
\rho^\gamma_{\mu\mu'}(i) = \frac{1}{4} G(i) R_{\mu\mu'}^\gamma(i),
\]

where

\[
R_{\mu\mu'}^\gamma(i) = 4 \sum_M \sum_{M'} c_M^* c_{M'} D_{M\mu}^1(\phi_k, \theta_k, -\phi_k)^* D_{M'\mu'}^1(\phi_k, \theta_k, -\phi_k)
\]

governs completely the angular dependence. The \( G(i) \) are given by

\[
G(i) = |\mathcal{J}|^2 p_i F(i,x)
\]

in terms of the profile function

\[
F(i,x) = |f(i)|^2 = \frac{1}{(E_i - E_0 - x\Gamma_i)^2 + \Gamma_i^2},
\]

where

\[
x = \frac{\omega - \omega_0}{\Gamma_i}, \quad \omega_0 = E_0 - E_i
\]

denotes the conventional frequency displacement from the line center in natural width units. It may also be noted that the frequency dependence of \( \rho^\gamma(i) \) as well as the line strengths are completely taken care of by \( G(i) \).

Defining the Fano statistical tensors \( t_q^k \) through

\[
t_q^k = \sqrt{3} \sum_{M=1}^{1} (-1)^{M-1} C(1,1,k; M', -M, q) \sum_{i=1}^{3} c_M^i p_i F(i,x) c_{M'}^i,
\]

we can also express Eq. (48) in another elegant form

\[
\rho^\gamma_{\mu\mu'} = \sum_{k=0}^{2} (t^k \cdot \mathcal{J}^k(\mu, \mu'))
\]

\[
= \sum_{k=0}^{2} \sum_{q=-k}^{k} (-1)^q t_q^k \mathcal{J}^q_{-q}(\mu, \mu'),
\]

where the analyzing powers are given by

\[
\mathcal{J}^q_{-q}(\mu, \mu') = \frac{1}{3} |\mathcal{J}|^2 \sum_{M=-1}^{1} (-1)^{1-M} C(1,1,k; M, -M, -q)
\]

\[
\times D_{M\mu}^1(\phi_k, \theta_k, -\phi_k)^* D_{M'\mu'}^1(\phi_k, \theta_k, -\phi_k).
\]
4.3. Stokes parameters

We may now express $\rho^i$ in the form [40,41]

$$\rho^i = \frac{1}{2}[I + \sigma \cdot S],$$

where $\sigma$ denotes Pauli matrices, with rows and columns labeled by the left and right circular polarization states $|\mu = \pm 1\rangle$ of the radiation. It follows that

$$I = Tr \rho^i = \sum_{\mu = -1,1} \rho^i_{\mu\mu},$$

denotes the intensity and $S$ denotes the well-known Stokes parameters [42] defined through the expressions

$$S_1 = Tr(\rho^i \sigma_x) = (\rho_{+1,-1} + \rho_{-1,+1}) = Q,$$

$$S_2 = Tr(\rho^i \sigma_y) = i(\rho_{+1,-1} - \rho_{-1,+1}) = U,$$

$$S_3 = Tr(\rho^i \sigma_z) = (\rho_{+1,+1} - \rho_{-1,-1}) = V,$$

where the notation $(I, Q, U, V)$ is more often used in the literature [42] to represent the Stokes vector.

5. Results and discussion

It is particularly interesting to make a detailed comparison of the emergent Stokes profiles for the spectral lines formed in the presence of a pure electric quadrupole field with those in the case of Zeeman effect.

5.1. Stokes line profiles formed in the presence of a pure magnetic field

The states $\Psi_l$ in this case are readily identified with $|1,M\rangle$, where $M = +1,0,-1$ defined with respect to the axis of quantization parallel to $B$. Note that the variation with the frequency in the vicinity of $E_M - E_l$ for each $E_M$ is governed by the profile function $F(M,x)$, which is shown in Fig. 4 with the peaks normalized to unity. The widths $\Gamma_M$ have been chosen appropriate to the Ca I line and set as $\Gamma = 2.18 \times 10^8$ s$^{-1}$ independent of $M$. We can now explicitly write the density matrices $\rho^i(M)$ for $M = +1,0,-1$ using Eq. (49), where we have

$$R^i_{+1,+1}(M = 1) = (1 + \cos \theta_k)^2$$

$$R^i_{+1,-1}(M = 1) = \sin^2 \theta_k e^{-2i\phi_k}$$

$$R^i_{-1,+1}(M = 1) = \sin^2 \theta_k e^{2i\phi_k}$$

$$R^i_{-1,-1}(M = 1) = (1 - \cos \theta_k)^2,$$

(62)
for the transition from the $M = +1$ state. Using Eq. (58)-(61), the Stokes parameters for this line component are given by

$$I(M = 1) = \frac{G(M = 1)}{2} (1 + \cos^2 \theta_k)$$

$$Q(M = 1) = \frac{G(M = 1)}{2} \sin^2 \theta_k \cos 2\phi_k$$

$$U(M = 1) = \frac{G(M = 1)}{2} \sin^2 \theta_k \sin 2\phi_k$$

$$V(M = 1) = G(M = 1) \cos \theta_k.$$  \hspace{1cm} (63)

Likewise, for the transition from $M = 0$ upper state, we have

$$R_{1,+1}^0(M = 0) = \sin^2 \theta_k,$$

$$R_{1,-1}^0(M = 0) = -\sin^2 \theta_k e^{-2i\phi_k},$$

$$R_{-1,+1}^0(M = 0) = -\sin^2 \theta_k e^{2i\phi_k},$$

$$R_{-1,-1}^0(M = 0) = \sin^2 \theta_k$$  \hspace{1cm} (64)

and the Stokes parameters are

$$I(M = 0) = G(M = 0) \sin^2 \theta_k,$$

$$Q(M = 0) = -G(M = 0) \sin^2 \theta_k \cos 2\phi_k,$$
\[ U(M = 0) = -G(M = 0) \sin^2 \theta_k\sin 2\phi_k, \]
\[ V(M = 0) = 0, \]

whereas for the transition from \(M = -1\) state, we have
\[ R^{2}_{1,1}(M = -1) = (1 - \cos \theta_k)^2, \]
\[ R^{2}_{1,-1}(M = -1) = \sin^2 \theta_k e^{-2i\phi_k}, \]
\[ R^{2}_{-1,1}(M = -1) = \sin^2 \theta_k e^{2i\phi_k}, \]
\[ R^{2}_{-1,-1}(M = -1) = (1 + \cos \theta_k)^2 \]

and the Stokes parameters
\[ I(M = -1) = \frac{1}{2} G(M = -1)(1 + \cos^2 \theta_k) \]
\[ Q(M = -1) = \frac{1}{2} G(M = -1) \sin^2 \theta_k \cos 2\phi_k \]
\[ U(M = -1) = \frac{1}{2} G(M = -1) \sin^2 \theta_k \sin 2\phi_k \]
\[ V(M = -1) = -G(M = -1) \cos \theta_k. \]

The above results are graphically presented in Fig. 5 for some select angles \(\theta_k = 0, \pi/4, \pi/2\) and \(\phi_k = 0\). Clearly the Zeeman components with \(M = \pm 1\) are 100\% circularly polarized and the \(M = 0\) component is absent at \(\theta_k = 0\) and \(\pi/4\), whereas at \(\theta_k = \pi/2\), the circular polarization asymmetry is zero for all the three components each of which is now linearly polarized. It is obvious that there should be no polarization for zero field, when the lines corresponding to the transitions from the \(\Psi_{+1}\) and \(\Psi_{-1}\) states occur at the same point \(x = 0\), with the same intensity, so that \(I = I(M = +1) + I(M = -1)\) increases, whereas \(V = V(M = +1) - V(M = -1)\) becomes zero. If there were no widths, even a slight deviation from \(x = 0\) on either side should be characterized by \(I = 0\). On the other hand \(I\) decreases slowly on either side of \(x = 0\) because of the finite width. The finite width is also responsible for the depolarization in \(V\) as the lines corresponding to \(M = +1\) and \(M = -1\) approach \(x = 0\) from either side, in the weak field case. While \(V\) being the difference in the intensities of \(M = +1\) and \(-1\) decreases and \(I\) being the sum of \(I = I(M = +1)\) and \(I(M = -1)\) remains sufficiently large so that the depolarization is pronounced when we plot \(V/I\) in Fig. 5.

5.2. Stokes line profiles formed in the presence of a pure electric quadrupole field

In this case, the relevant eigenstates \(\Psi_i, i = x, y, z\) of the atomic density matrix \(\rho\) are given by Eq. (19) in Section 3 and the respective energy level separations are shown in Fig. 2. The profile functions for the pure electric quadrupole case are shown in Figs. 6(a)–(c) with peaks normalized
Fig. 5. The effect of pure magnetic field. Panels (a)–(c) represent the emission Stokes line profiles of Ca I line for assumed temperature value $T = 6000K$ and natural width $\Gamma = 2.18 \times 10^8 s^{-1}$. The three different curves (solid, dash dot dot and dashed) correspond to the ratio of level splitting to natural width $= 0.4, 4, 8$ respectively. The Stokes $Q$ and $V$ are expressed in units of intensity. $x = \omega - \omega_0$ is the frequency displacement from the line center in natural width units.

to unity and with a choice of $\Gamma = 2.18 \times 10^8 s^{-1}$ independent of $i$, for illustrative purposes. It may be emphasized that the profile functions for the pure electric field are characteristically different from those for the pure magnetic field. Moreover they are dependent also on the asymmetry parameter $\eta$. 
Fig. 6. The profile function $F(i, x)$ showing the pure electric quadrupole field effect. Panels (a)–(c) represent the asymmetry parameter $\eta = 0, 0.5, 1$. The solid, dotted and dashed lines correspond to the excited states $\Psi_x, \Psi_y, \Psi_z$ respectively. $x$ is the frequency displacement from the line center in natural width units.

For $i=x$ the elements of $\rho^i(x)$ are explicitly given in terms of the photon emission angles ($\theta_k, \phi_k$) by

\begin{align}
R_{+1,+1}^i(x) &= (1 + \cos^2 \theta_k) - \sin^2 \theta_k \cos 2\phi_k, \\
R_{+1,-1}^i(x) &= \sin^2 \theta_k e^{-2i\phi_k} - 2^{-1}(1 - \cos \theta_k)^2 e^{-4i\phi_k} - 2^{-1}(1 + \cos \theta_k)^2, \\
R_{-1,+1}^i(x) &= \sin^2 \theta_k e^{2i\phi_k} - 2^{-1}(1 - \cos \theta_k)^2 e^{4i\phi_k} - 2^{-1}(1 + \cos \theta_k)^2, \\
R_{-1,-1}^i(x) &= (1 + \cos^2 \theta_k) - \sin^2 \theta_k \cos 2\phi_k.
\end{align}

(68)
From the above expressions, we obtain the emitted Stokes parameters

\[
I(x) = \frac{G(x)}{2} [(1 + \cos^2 \theta_k) - \sin^2 \theta_k \cos 2\phi_k],
\]

\[
Q(x) = \frac{G(x)}{2} [\sin^2 \theta_k \cos 2\phi_k - 2^{-1} (1 - \cos \theta_k)^2 \cos 4\phi_k - 2^{-1} (1 + \cos \theta_k)^2],
\]

\[
U(x) = \frac{G(x)}{2} [\sin^2 \theta_k \sin 2\phi_k - 2^{-1} (1 - \cos \theta_k)^2 \sin 4\phi_k],
\]

\[
V(x) = 0.
\]

Similarly, the matrix elements \( R^{\mu\nu}_{y}(y) \) are given by

\[
R^{+1,+1}_{1,-1}(y) = (1 + \cos^2 \theta_k) + \sin^2 \theta_k \cos 2\phi_k,
\]

\[
R^{-1,-1}_{1,+1}(y) = \sin^2 \theta_k e^{-2i\phi_k} + 2^{-1} (1 - \cos \theta_k)^2 e^{-4i\phi_k} + 2^{-1} (1 + \cos \theta_k)^2,
\]

\[
R^{-1,+1}_{-1,-1}(y) = (1 + \cos^2 \theta_k) + \sin^2 \theta_k \cos 2\phi_k.
\]

The corresponding expressions for the Stokes parameters are

\[
I(y) = \frac{G(y)}{2} [(1 + \cos^2 \theta_k) + \sin^2 \theta_k \cos 2\phi_k],
\]

\[
Q(y) = \frac{G(y)}{2} [\sin^2 \theta_k \cos 2\phi_k + 2^{-1} (1 - \cos \theta_k)^2 \cos 4\phi_k + 2^{-1} (1 + \cos \theta_k)^2],
\]

\[
U(y) = \frac{G(y)}{2} [\sin^2 \theta_k \sin 2\phi_k + 2^{-1} (1 - \cos \theta_k)^2 \sin 4\phi_k],
\]

\[
V(y) = 0.
\]

For \( \rho^{\mu\nu}(z) \), we note that the angular dependence given by \( R^{\mu\nu}_{y}(z) \) is identical to that given by Eq. (64), since \( \Psi_z = \Psi_{1,0} \). However, the frequency dependence is governed by \( G(z) \), which is characteristically different from \( G(M = 0) \) since the energy eigenvalue \( E_z \) is different from \( E_0 \) and the spectrum of pure electric quadrupole field (dependent on \( \eta \) in addition to \( A \)) is different from the spectrum in the case of Zeeman effect. This results in non-symmetric Stokes profiles, unlike in the case of the Zeeman-effect. Thus, the Stokes parameters are given by

\[
I(z) = G(z) \sin^2 \theta_k,
\]

\[
Q(z) = -G(z) \sin^2 \theta_k \cos 2\phi_k,
\]

\[
U(z) = -G(z) \sin^2 \theta_k \sin 2\phi_k,
\]

\[
V(z) = 0.
\]
Fig. 7. The effect of pure electric quadrupole field ($\eta = 0$). Panels (a)-(c) represent the emission Stokes line profiles of Ca I line for assumed temperature value $T = 6000 K$ and natural width $\Gamma = 2.18 \times 10^8$ s$^{-1}$. The three different curves (solid, dash dot dot and dashed) correspond to the ratio of level splitting to natural width = 0.4, 0.8 respectively. The Stokes $Q$ is expressed in unit of intensity. $x = (\omega - \omega_0)/\Gamma$ is the frequency displacement from the line center in natural width units.

$\eta \neq 0$. Appearance of linear polarization $Q/I$ in the $Z$-direction of the PAF, is a remarkable feature observed in this case. Moreover, the $V$ parameter is absent, since the superposition $\Psi(M = +1)$ and $\Psi(M = -1)$ results in producing only linear polarization.

5.3. Stokes line profiles formed in the presence of both magnetic and electric quadrupole fields

We consider in this paper only the simple case of magnetic field $B$ along the $Z$-axis of the PAF. The corresponding eigenstates $\Psi_i$, $i = 1, 2, 3$ of the atomic density matrix $\rho$ are therefore given by
Fig. 8. The effect of pure electric quadrupole field ($\eta = 0.5$). Panels (a)-(c) represent the emission Stokes line profiles of Ca I line for assumed temperature value $T = 6000 \, K$ and natural width $\Gamma = 2.18 \times 10^8 \, s^{-1}$. The three different curves (solid, dash dot dot and dashed) correspond to the ratio of level splitting to natural width $= 0.4, 4, 8$ respectively. The Stokes $Q$ is expressed in unit of intensity. $x = \frac{\omega - \omega_0}{\Gamma}$ is the frequency displacement from the line center in natural width units.

Eq. (23) in Section 3.3 and the respective energy level separations, which depend on the magnetic field strength and the electric quadrupole field parameters $A$ and $\eta$, are shown in Fig. 3. In this case, the level separations are unequal, except when $(A^2\eta^2 + D^2)^{1/2} = A$. The angular dependence in the elements of $\rho^{(1)}$ is the same as in Eq. (65), since the state $\Psi_1$ is identical to $\Psi_{1,0}$, but the frequency dependence is characteristically different since it is governed by $G(1)$. It is important to note that $G(M = 0)$, $G(z)$ and $G(1)$ are responsible for the remarkable changes in the Stokes line profiles considered here in the three cases of pure magnetic field, pure electric quadrupole field
Fig. 9. The effect of pure electric quadrupole field ($\eta = 1$). Panels (a)–(c) represent the emission Stokes line profiles of Ca I line for assumed temperature value $T = 6000\,K$ and natural width $\Gamma = 2.18 \times 10^8\,s^{-1}$. The three different curves (solid, dash dot dot and dashed) correspond to the ratio of level splitting to natural width $= 0.4, 4, 8$ respectively. The Stokes $Q$ is expressed in unit of intensity. $x = \frac{\omega - \omega_0}{\Gamma}$ is the frequency displacement from the line center in natural width units.

and the combination of electric and magnetic fields respectively although the factors responsible for angular distribution are identical. The stokes parameters here are given by

$$
I(1) = G(1) \sin^2 \theta_k,
$$

$$
Q(1) = -G(1) \sin^2 \theta_k \cos 2\phi_k,
$$

$$
U(1) = -G(1) \sin^2 \theta_k \sin 2\phi_k,
$$

$$
V(1) = 0.
$$

(73)
For the transition from the state $\Phi_2$, the $R_{\mu'\mu}(2)$ are given by

\[
R_{+1,+1}(2) = a^2(1 - \cos \theta_k)^2 + 2ab \sin^2 \theta_k \cos 2\phi_k + b^2(1 + \cos \theta_k)^2,
\]
\[
R_{+1,-1}(2) = (a^2 + b^2) \sin^2 \theta_k e^{-2i\phi_k} + ab(1 - \cos \theta_k)^2 e^{-4i\phi_k} + ab(1 + \cos \theta_k)^2,
\]
\[
R_{-1,+1}(2) = (a^2 + b^2) \sin^2 \theta_k e^{2i\phi_k} + ab(1 - \cos \theta_k)^2 e^{4i\phi_k} + ab(1 + \cos \theta_k)^2,
\]
\[
R_{-1,-1}(2) = a^2(1 + \cos \theta_k)^2 + 2ab \sin^2 \theta_k \cos 2\phi_k + b^2(1 - \cos \theta_k)^2. \quad (74)
\]

The corresponding Stokes parameters for this transition are

\[
I(2) = G(2)[\frac{1}{2} (a^2 + b^2)(1 + \cos^2 \theta_k) + ab \sin^2 \theta_k \cos 2\phi_k],
\]
\[
Q(2) = \frac{G(2)}{2} \left[ (a^2 + b^2) \sin^2 \theta_k \cos 2\phi_k + ab(1 - \cos \theta_k)^2 \cos 4\phi_k + ab(1 + \cos \theta_k)^2 \right],
\]
\[
U(2) = \frac{G(2)}{2} \left[ (a^2 + b^2) \sin^2 \theta_k \sin 2\phi_k + ab(1 - \cos \theta_k)^2 \sin 4\phi_k \right],
\]
\[
V(2) = G(2)(a^2 - b^2) \cos \theta_k. \quad (75)
\]

For the transition from the state $\Phi_3$, likewise, we have

\[
R_{+1,+1}(3) = a^2(1 + \cos \theta_k)^2 - 2ab \sin^2 \theta_k \cos 2\phi_k + b^2(1 - \cos \theta_k)^2,
\]
\[
R_{+1,-1}(3) = (a^2 + b^2) \sin^2 \theta_k e^{-2i\phi_k} - ab(1 - \cos \theta_k)^2 e^{-4i\phi_k} - ab(1 + \cos \theta_k)^2,
\]
\[
R_{-1,+1}(3) = (a^2 + b^2) \sin^2 \theta_k e^{2i\phi_k} - ab(1 - \cos \theta_k)^2 e^{4i\phi_k} - ab(1 + \cos \theta_k)^2,
\]
\[
R_{-1,-1}(3) = a^2(1 - \cos \theta_k)^2 - 2ab \sin^2 \theta_k \cos 2\phi_k + b^2(1 + \cos \theta_k)^2. \quad (76)
\]

The Stokes parameters for this transition are given by

\[
I(3) = G(3)[\frac{1}{2} (a^2 + b^2)(1 + \cos^2 \theta_k) - ab \sin^2 \theta_k \cos 2\phi_k],
\]
\[
Q(3) = \frac{G(3)}{2} \left[ (a^2 + b^2) \sin^2 \theta_k \cos 2\phi_k - ab(1 - \cos \theta_k)^2 \cos 4\phi_k - ab(1 + \cos \theta_k)^2 \right],
\]
\[
U(3) = \frac{G(3)}{2} \left[ (a^2 + b^2) \sin^2 \theta_k \sin 2\phi_k - ab(1 - \cos \theta_k)^2 \sin 4\phi_k \right],
\]
\[
V(3) = G(3)(a^2 - b^2) \cos \theta_k. \quad (77)
\]

The above results are presented in Figs. 10–12 for different values of the asymmetry parameter $\eta = 0, 0.5$ and 1 respectively for varying ratios of magnetic field to electric field strengths. This case, i.e. where the radiating atom is in the presence of both an electric quadrupole field and a magnetic field, presents a host of interesting features. As is evident from the figures, we tend toward the Zeeman-case when the magnetic field becomes stronger. It is worth drawing one’s attention to the $V$ parameter in these cases, a parameter not present in the pure electric quadrupole field case.
Fig. 10. The effect of combined magnetic and electric quadrupole fields on Stokes line profiles for \( \eta = 0 \). Panels (a)-(c) represent the emission Stokes line profiles of Ca I line for assumed temperature value \( T = 6000 \) K and natural width \( \Gamma = 2.18 \times 10^8 \) s\(^{-1}\). The three different curves (solid, dash dot dot and dashed) correspond to the ratio of magnetic to electric fields strengths =1, 3, 5 respectively. The Stokes \( Q \) and \( V \) are expressed in units of intensity. \( x = \frac{\nu - \nu_0}{\Gamma} \) is the frequency displacement from the line center in natural width units.
Fig. 11. The effect of combined magnetic and electric quadrupole fields on Stokes line profiles for $\eta = 0.5$. Panels (a)-(c) represent the emission Stokes line profiles of Ca I line for assumed temperature value $T = 6000\, K$ and natural width $\Gamma = 2.18 \times 10^8\, s^{-1}$. The three different curves (solid, dash dot dot and dashed) correspond to the ratio of magnetic to electric fields strengths =1,3,5 respectively. The Stokes $Q$ and $V$ are expressed in units of intensity. $x = \frac{\omega - \omega_0}{\Gamma}$ is the frequency displacement from the line center in natural width units.
Fig. 12. The effect of combined magnetic and electric quadrupole fields on Stokes line profiles for $\eta = 1$. Panels (a)-(c) represent the emission Stokes line profiles of Ca I line for assumed temperature value $T = 6000 K$ and natural width $\Gamma = 2.18 \times 10^8 \text{ s}^{-1}$. The three different curves (solid, dash dot dot and dashed) correspond to the ratio of magnetic to electric fields strengths = 1, 3, 5 respectively. The Stokes $Q$ and $V$ are expressed in units of intensity. $x = \frac{\omega - \omega_0}{\Gamma}$ is the frequency displacement from the line center in natural width units.
We notice a modification in the shape of this profile near line center, due to the presence of the electric quadrupole field. Thus, a careful examination of the shape of the $V$ profile near line center, can serve as a diagnostic for the presence electric quadrupole field in addition to the magnetic field.

5.4. Possible applications

The theoretical results presented here could be expected to be useful to study the electric fields in meteorological and cloud physics, since it is very well known that accumulated electric charges are responsible for the phenomenon of lightning. It may therefore be advantageous to study the polarized line profiles of lightning keeping in view the theoretical formulae derived herein. Our results may also be used to detect the electric fields in solar and stellar atmospheres, by modeling the polarized line profiles formed in these atmospheric layers. The theoretical predictions may also be verified experimentally through laboratory studies on neutral atoms in suitably created environments like that of Paul trap in the case of pure electric quadrupole field and Penning trap in the case of combined electric quadrupole field and uniform magnetic field.

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